

DERIVED ISOGENIES AND ISOGENIES FOR ABELIAN SURFACES

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ABSTRACT. In this paper, we study the twisted Fourier-Mukai partners of abelian surfaces. Following the work of Huybrechts [31], we introduce the twisted derived equivalences (also called derived isogenies) between abelian surfaces. We show that there is a twisted derived Torelli theorem for abelian surfaces over algebraically closed fields with characteristic $\neq 2, 3$. Over the complex numbers, the derived isogenies correspond to rational Hodge isometries between the second cohomology groups, which is in analogy to the work of Huybrechts and Fu–Vial on K3 surfaces. Their proof relies on the global Torelli theorem over \mathbb{C} , which is missing in positive characteristics. To overcome this issue, we firstly extend a trick given by Shioda on integral Hodge structures, to rational Hodge structures, ℓ -adic Tate modules and F -crystals. Then we make use of Tate’s isogeny theorem to give a characterization of the derived isogenies between abelian surfaces via so called principal isogenies. As a consequence, we show the two abelian surfaces are principally isogenous if and only if they are derived isogenous.

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1. INTRODUCTION

1.1. **Background.** In the study of abelian varieties, a natural question is to classify the Fourier-Mukai partners of abelian varieties. Due to Orlov and Polishchuk’s *derived Torelli theorem* for abelian varieties in (*cf.* [52, 54]), there is a geometric/cohomological classification of derived equivalences between them. More generally, one can consider the *twisted derived equivalence* or so called *derived isogeny* between abelian varieties in the spirit of [31]: two abelian varieties X and Y are derived isogenous if they can be connected by derived equivalences between twisted abelian varieties, i.e. there exist twisted abelian varieties (X_i, α_i) and (X_i, β_i) such that there is a zig-zag of derived equivalences

$$\begin{aligned} D^b(X, \alpha) &\xrightarrow{\simeq} D^b(X_1, \beta_1) \\ D^b(X_1, \alpha_2) &\xrightarrow{\simeq} D^b(X_2, \beta_2) \\ &\vdots \\ D^b(X_n, \alpha_{n+1}) &\xrightarrow{\simeq} D^b(Y, \beta_n) \end{aligned} \tag{1.1.1}$$

where $D^b(X, \alpha)$ is the bounded derived category of α -twisted coherent sheaves on X .

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In [60], Stellari proved that derived isogenous complex abelian surfaces are isogenous using the the Kuga–Satake varieties associated to their transcendental lattices (*cf.* Theorem 1.2 in *loc.cit.*). However, the converse is not true as there are isogenous abelian surfaces which are not derived isogenous (*cf.* Remark 4.4 (ii) in *loc.cit.*). The main goal of this paper is to give a cohomological and geometric classification of derived isogenies between abelian surfaces over algebraically closed fields of arbitrary characteristic.

1.2. Twisted derived Torelli theorem for abelian surfaces in characteristic zero. Let us first classify the derived isogenous between abelian surfaces in term of isogenies. For this purpose, we need to introduce a new type of isogeny. We say two abelian surfaces X and Y are *principally isogenous* if there is a isogeny f from X or \widehat{X} to Y of square degree. The first main result is

Theorem 1.2.1. *Let X and Y be two abelian surfaces over $k = \bar{k}$ with $\text{char}(k) = 0$. The following statements are equivalent.*

- (i) X and Y are derived isogenous.
- (ii) X and Y are principally isogenous.

A notable fact for abelian surfaces is that besides their 1st cohomology groups, their 2nd cohomology groups also carry rich structures. In the untwisted case, Mukai and Orlov have showed [48, 52] that

$$D^b(X) \cong D^b(Y) \Leftrightarrow \widetilde{H}(X, \mathbb{Z}) \cong_{\text{Hdg}} \widetilde{H}(Y, \mathbb{Z}) \Leftrightarrow T(X) \cong_{\text{Hdg}} T(Y),$$

where $\widetilde{H}(X, \mathbb{Z})$ and $\widetilde{H}(Y, \mathbb{Z})$ are the Mukai lattices, $T(X) \subseteq H^2(X, \mathbb{Z})$ and $T(Y) \subseteq H^2(Y, \mathbb{Z})$ denote the transcendental lattices, \cong_{Hdg} means integral Hodge isometries (*cf.* [12, Theorem 5.1]). The following result can be viewed as a generalization of Mukai and Orlov’s result.

Corollary 1.2.2. *The statement (i) is also equivalent to the following equivalent conditions*

- (iii) *the associated Kummer surfaces $\text{Km}(X)$ and $\text{Km}(Y)$ are derived isogenous;*
- (iv) *Chow motives $\mathfrak{h}(X) \cong \mathfrak{h}(Y)$ are isomorphic as Frobenius exterior algebras;*
- (v) *even degree Chow motives $\mathfrak{h}^{\text{even}}(X) \cong \mathfrak{h}^{\text{even}}(Y)$ are isomorphic as Frobenius algebra.*

When $k = \mathbb{C}$, then the conditions above are also equivalent to

- (vi) $H^2(X, \mathbb{Q}) \cong H^2(Y, \mathbb{Q})$ as a rational Hodge isometry;
- (vii) $\widetilde{H}(X, \mathbb{Q}) \cong \widetilde{H}(Y, \mathbb{Q})$ as a rational Hodge isometry;
- (viii) $T(X) \otimes \mathbb{Q} \cong T(Y) \otimes \mathbb{Q}$ as a rational Hodge isometry.

Here, the motive $\mathfrak{h}(X)$ admits a canonical motivic decomposition produced by Deninger–Murre [19]

$$\mathfrak{h}(X) = \bigoplus_{i=0}^4 \mathfrak{h}^i(X) \tag{1.2.1}$$

such that $H^*(\mathfrak{h}^i(X)) \cong H^i(X)$ for any Weil cohomology $H^*(-)$. It satisfies $\mathfrak{h}^i(X) = \bigwedge^i \mathfrak{h}^1(X)$ for all i , $\mathfrak{h}^4(X) \simeq \mathbb{1}(-4)$ and $\bigwedge^i \mathfrak{h}^1(X) = 0$ for $i > 4$ (*cf.* [37]). The motive $\mathfrak{h}(X)$ is a Frobenius exterior algebra objects in the category of Chow motives over k and the even degree part

$$\mathfrak{h}^{\text{even}}(X) = \bigoplus_{k \geq 0}^2 \bigwedge^{2k} \mathfrak{h}^1(X) \tag{1.2.2}$$

forms a Frobenius algebra object in the sense of [23].

The equivalences (i) \Leftrightarrow (iv) \Leftrightarrow (v) are motivic realizations of derived isogenies between abelian surfaces, which can be viewed as an analogy of the motivic global Torelli theorem on K3 surfaces (*cf.* [31, Conjecture 0.3] and [23, Theorem 1]). The equivalences (i) \Leftrightarrow (iii) \Leftrightarrow (viii) can be viewed as a generalization of [60, Theorem 1.2]. The Hodge-theoretic realization (i) \Leftrightarrow (vi) follows a similar strategy of [31, Theorem 0.1], which makes use of Shioda’s period map and Cartan–Dieudonné decomposition of a rational isometry. The equivalences (vi) \Leftrightarrow (vii) \Leftrightarrow (viii) follow from the Witt cancellation theorem (see §5.3).

1.3. Shioda's trick. The proof of Theorem 1.2.1 is concluded by a new ingredient so called rational Shioda's trick on abelian surfaces. The original Shioda's trick in [58] plays a key role in the proof of Shioda's global Torelli theorem for abelian surfaces, which links the weight-1 integral Hodge structure to the weight-2 integral Hodge structure of an abelian surface. We generalize it in the following form.

Theorem 1.3.1 (Shioda's trick, see §4). *Let X and Y be two complex abelian surfaces. Then for any admissible Hodge isometry*

$$\psi: H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q})$$

we can find an isogeny $f: Y \rightarrow X$ of degree d^2 such that $\psi = \frac{f^}{d}$.*

As an application, the generalized Shioda's trick gives the algebraicity of some cohomological cycles. For any integer d , one can consider a Hodge similitude of degree d

$$H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q}),$$

called a *Hodge isogeny of degree d* . Due to the Hodge conjecture on products of abelian surfaces, we know that every Hodge isogeny is algebraic. Our generalized Shioda's trick actually shows that it is induced by certain isogenies. Similarly, we prove the ℓ -adic and p -adic Shioda's trick, which gives a proof of Tate conjecture for isometries between the 2^{nd} -cohomology groups (as either Galois-modules or crystals) of abelian surfaces over finitely generated fields. See Corollary 4.6.4 for more details.

1.4. Results in positive characteristic. The second part of this paper is to investigate the twisted derived Torelli theorem over positive characteristic fields. Due to the absence of a satisfactory global Torelli theorem, one can not follow the argument in characteristic zero directly. Instead, we need some input from p -adic Hodge theory. Our formulation is the following.

Theorem 1.4.1. *Let X and Y be two abelian surfaces over $k = \bar{k}$ with $\text{char}(k) = p > 3$. Then the following statements are equivalent.*

- (i') *X and Y are prime-to- p derived isogenous.*
- (ii') *X and Y are prime-to- p principally isogenous.*

Moreover, in case that X is supersingular, then Y is derived isogenous to X if and only if Y is supersingular.

Here, we say a derived isogeny as (1.1.1) is *prime-to- p* if its crystalline realization is integral (see Definition 3.1.3 for details), which is a condition somewhat technical. The main ingredients in the proof of Theorem 1.4.1 are the lifting-specialization technique, which works well for prime-to- p derived isogenies. Actually, our method shows that there is an implication (i') \Rightarrow (ii') for derived isogenies which are not necessarily being prime-to- p (see Theorem 6.5.1). Conversely, we believe that the existence of quasi-liftable isogenies will imply the existence of derived isogeny (see Conjecture 6.5.2). The only obstruction is the existence of the specialization of non-prime-to- p derived isogenies between abelian surfaces. See Remark 6.3.3.

Another natural question is whether two abelian surfaces are derived isogenous if and only if their associated Kummer surfaces are derived isogenous over positive characteristic fields. Unfortunately, we can not fully prove the equivalence. Instead, we provide a partial solution of this question. See Theorem 6.5.3 for more details.

Similarly, one may ask whether such results also hold for K3 surfaces. Recall that two K3 surfaces S and S' over a finite field \mathbb{F}_q are (geometrically) isogenous in the sense of [65] if there exists an algebraic correspondence Γ which induces an isometry of $\text{Gal}(\bar{\mathbb{F}}_p/k)$ -modules

$$\Gamma_\ell^*: H_{\text{ét}}^2(S_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^2(S'_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell),$$

for all $\ell \nmid p$ and an isometry of isocrystals

$$\Gamma_p^*: H_{\text{crys}}^2(S_k/K) \xrightarrow{\sim} H_{\text{crys}}^2(S'_k/K),$$

for some finite extension k/\mathbb{F}_q and the fraction field K of $W = W(k)$. Then we say the isogeny is prime-to- p if the isometry Γ_p^* is integral, i.e., $\Gamma_p^*(\mathbf{H}_{\text{crys}}^2(S_k/W)) = \mathbf{H}_{\text{crys}}^2(S'_k/W)$. Then we have a formulation of the twisted derived Torelli conjecture for K3 surfaces.

Conjecture 1.4.2. *For two K3 surfaces S and S' over a finite field k with $\text{char}(k) = p > 0$, then the following are equivalent.*

- (a) *There exists a prime-to- p derived isogeny $\mathbf{D}^b(S) \sim \mathbf{D}^b(Y)$.*
- (b) *There exists a prime-to- p isogeny between S and S' .*

The implication (a) \Rightarrow (b) is clear, while the converse remains open. In the case of Kummer surfaces, our results provide some evidence of Conjecture 1.4.2. We shall mention that recently Bragg and Yang have studied the derived isogenies between K3 surfaces over positive characteristic fields and they provided a weaker version of the statement in Conjecture 1.4.2 (cf. [9, Theorem 1.2]).

Organization of the paper. We will start with two preliminary sections, in which we include some well-known constructions and facts: In Section 2, we perform the computations of the Brauer group of abelian surfaces via the Kummer construction. This allows us to prove the lifting lemma for twisted abelian surfaces of finite height. In Section 3, we collect the knowledge on derived isogenies between abelian surfaces and their cohomological realizations, which include the motivic realization, the \mathbf{B} -field theory, the twisted Mukai lattices, a filtered Torelli theorem and its relation to the moduli space of twisted sheaves.

In Section 4, we revise Shioda's work and extend it to rational Hodge isogenies. This is the key ingredient for proving Theorem 1.2.1. Furthermore, after introducing the admissible ℓ -adic and p -adic bases, we prove the ℓ -adic and p -adic Shioda's trick for admissible isometries on abelian surfaces. As an application, we prove the algebraicity of these isometries on abelian surfaces over finitely generated fields.

Section 5 and 6 are devoted to proving Theorem 1.2.1 and Theorem 1.4.1. Theorem 1.2.1 is essentially Theorem 5.1.3 and Theorem 5.2.5. The proof of Theorem 1.4.1 is much more subtle. We establish the lifting and the specialization theorem for prime-to- p derived isogeny. Then one can conclude (i') \Leftrightarrow (ii') from Theorem 1.2.1 for abelian surfaces of finite heights. At the end of Section 6, we follow Bragg and Lieblich's twistor line argument in [7] to conclude the supersingular case of Theorem 1.4.1.

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Notations and Conventions. Throughout this paper, we will use the symbol k to denote a field. If k is a perfect field and $\text{char } k = p > 0$, we denote $W := W(k)$ for the ring of Witt vectors in k , which is equipped with a morphism $\sigma: W \rightarrow W$ induced by the Frobenius map on k . If k is not perfect, we choose a Cohen ring $W \subset W(\bar{k})$ with $W/pW = k$, inside the ring of Witt vectors in a fixed algebraic closure \bar{k} of k .

Let X be a smooth projective variety over k . We denote by $\mathbf{H}_{\text{ét}}^\bullet(X_{\bar{k}}, \mathbb{Z}_\ell)$ the ℓ -adic étale cohomology group of $X_{\bar{k}}$. The \mathbb{Z}_ℓ -module $\mathbf{H}_{\text{ét}}^\bullet(X_{\bar{k}}, \mathbb{Z}_\ell)$ has been endowed with a canonical $G_k = \text{Gal}(\bar{k}/k)$ -action. We use $\mathbf{H}_{\text{crys}}^i(X/W)$ to denote the i -th crystalline cohomology group of X over the p -adic base $W \rightarrow k$, which is a W -module. It is endowed with a natural σ -linear map

$$\varphi: \mathbf{H}_{\text{crys}}^i(X/W) \rightarrow \mathbf{H}_{\text{crys}}^i(X/W)$$

induced from the absolute Frobenius morphism $F_X: X \rightarrow X$.

We denote by $\mathbf{D}^b(X)$ the bounded derived category of coherent sheaves X . A derived equivalence means a k -linear exact equivalence between triangulated categories in the form

$$\Psi: \mathbf{D}^b(X) \xrightarrow{\sim} \mathbf{D}^b(Y).$$

If Ψ is of the form

$$\Psi^P(E) = \mathbf{R}q_*(p^*E \otimes \mathcal{P}),$$

then we call it a Fourier–Mukai transform with a kernel $P \in \mathbf{D}^b(X \times Y)$ and the projections $p: X \times Y \rightarrow X$, $q: X \times Y \rightarrow Y$, and X, Y are called a pair of Fourier–Mukai partners.

When X is an abelian variety over k , we denote \widehat{X} for its dual abelian variety and $X[p^\infty]$ for the associated p -divisible group. There is a natural identification of its contravariant Dieudonné module with its first crystalline cohomology:

$$\mathbb{D}(X[p^\infty]) := M(X[p^\infty]^\vee) \cong H_{\text{crys}}^1(X/W),$$

where $M(-)$ is the Dieudonné module functor on p -divisible groups defined in [44, 17].

For any abelian group G and an integer n , we denote $G[n]$ for the subgroup of n -torsions in G and $G\{n\}$ for the union of all n -power torsions.

For a lattice L in \mathbb{Z} or \mathbb{Q} and an integer n , we use $L(n)$ for the lattice twisted by n , i.e., $L = L(n)$ as \mathbb{Z} or \mathbb{Q} -module, but

$$\langle x, y \rangle_{L(n)} = n \langle x, y \rangle_L.$$

The reader shall not confuse it with the Tate twist.

2. TWISTED ABELIAN SURFACE

In this section, we give some preliminary results in the theory of twisted abelian surfaces, especially for those in positive characteristic. As most of these results are well-known in characteristic zero, the readers who are only interested in this case may skip this part.

We will frequently use the terminology “gerbe”, on which the readers may refer [25] or [39] for more details.

2.1. Gerbes on abelian surfaces and associated Kummer surfaces. Let X be an abelian surface over a field k and let $\mathcal{X} \rightarrow X$ be a μ_n -gerbe over X . This corresponds to a pair (X, α) for some $\alpha \in H_{\text{fl}}^2(X, \mu_n)$, where the cohomology group is with respect to the flat topology. Since μ_n is commutative, there is a bijection of sets

$$\{\mu_n\text{-gerbes on } X\} / \simeq \xrightarrow{\sim} H_{\text{fl}}^2(X, \mu_n),$$

where \simeq is the μ_n -equivalence defined as in [25, IV.3.1.1]. We may write $\alpha = [\mathcal{X}]$. The Kummer exact sequence induces a surjective map

$$H_{\text{fl}}^2(X, \mu_n) \rightarrow \text{Br}(X)[n] \tag{2.1.1}$$

where the right-hand side is the *cohomological Brauer group* $\text{Br}(X) := H_{\text{ét}}^2(X, \mathbb{G}_m)$. For any μ_n gerbe \mathcal{X} on X , there is an associated \mathbb{G}_m -gerbe on X via (2.1.1), denoted by $\mathcal{X}_{\mathbb{G}_m}$. Let $\mathcal{X}^{(m)}$ be the gerbe corresponding to cohomological class $m[\mathcal{X}] \in H_{\text{fl}}^2(X, \mu_n)$. If $[\mathcal{X}_{\mathbb{G}_m}] = 0$, then we will call \mathcal{X} an essentially-trivial μ_n -gerbe.

If k has characteristic $p \neq 2$, there is an associated Kummer surface \widetilde{X} constructed as follows:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\sigma}} & X \\ \downarrow \pi & & \downarrow \\ \text{Km}(X) & \xrightarrow{\sigma} & X/\iota \end{array} \tag{2.1.2}$$

where

- ι is the involution of X ;
- σ is the crepant resolution of quotient singularities;
- $\tilde{\sigma}$ is the blow-up of X along the closed subscheme $X[2] \subset X$. Its birational inverse is denoted by $\tilde{\sigma}^{-1}$.

Let $E \subset \widetilde{X}$ be the exceptional locus of $\tilde{\sigma}$. Then we have a composition of the sequence of morphisms

$$(\tilde{\sigma}^{-1})^* : \text{Br}(\widetilde{X}) \rightarrow \text{Br}(\widetilde{X} \setminus E) \cong \text{Br}(X \setminus X[2]) \cong \text{Br}(X).$$

Here, the last isomorphism $\text{Br}(X) \rightarrow \text{Br}(X \setminus X[2])$ is due to Grothendieck’s purity theorem (cf. [27, 63]).

Proposition 2.1.1. *When $k = \bar{k}$ and $\text{char}(k) \neq 2$, the $(\tilde{\sigma}^{-1})^* \pi^*$ induces an isomorphism between cohomological Brauer groups*

$$\Theta: \text{Br}(\text{Km}(X)) \rightarrow \text{Br}(X). \quad (2.1.3)$$

In particular, when X is supersingular over \bar{k} , then $\text{Br}(X)$ is isomorphic to the additive group \bar{k} .

Proof. For torsions of (2.1.3) whose orders are coprime to p , the proof is essentially the same as [59, Proposition 1.3] by the Hochschild–Serre spectral sequence and the fact that $H^2(\mathbb{Z}/2\mathbb{Z}, k^*) = 0$ (cf. [64, Proposition 6.1.10]) as $\text{char}(k) > 2$. See also [60, Lemma 4.1] for the case $k = \mathbb{C}$. For p -primary torsion part, we have

$$\text{Br}(\text{Km}(X))\{p\} \cong \text{Br}(X)^\iota\{p\}$$

from the Hochschild–Serre spectral sequence, where $\text{Br}(X)^\iota$ is the ι -invariant subgroup. Hence it suffices to prove that ι acts trivially on $\text{Br}(X)$.

In fact, $H_{\text{dR}}^2(X, \mu_p)$ can be ι -equivariantly embedded to $H_{\text{dR}}^2(X/k)$ by de Rham–Witt theory (cf. [50, Proposition 1.2]). The action of ι on $H_{\text{dR}}^2(X/k) = \wedge^2 H_{\text{dR}}^1(X/k)$ is the identity, as its action on $H_{\text{dR}}^1(X/k)$ is given by $x \mapsto -x$. Thus the involution on $H_{\text{dR}}^2(X, \mu_p)$ is trivial. Then by the exact sequence

$$0 \rightarrow \text{NS}(X) \otimes \mathbb{Z}/p \rightarrow H_{\text{dR}}^2(X, \mu_p) \rightarrow \text{Br}(X)[p] \rightarrow 0,$$

we can deduce that $\text{Br}(X)[p]$ is invariant under the involution. Furthermore, for p^n -torsions with $n \geq 2$, we can proceed by induction on n . Assume that all elements in $\text{Br}(X)[p^d]$ are ι -invariant if $1 \leq d < n$. By abuse of notation, we still use ι to denote the induced map $\text{Br}(X) \rightarrow \text{Br}(X)$. For $\alpha \in \text{Br}(X)[p^n]$, $p\alpha \in \text{Br}(X)[p^{n-1}]$ is ι -invariant. This gives

$$p\alpha = \iota(p\alpha) = p\iota(\alpha),$$

which implies $\alpha - \iota(\alpha) \in \text{Br}(X)[p]$. Applying ι on $\alpha - \iota(\alpha)$, we can obtain

$$\alpha - \iota(\alpha) = \iota(\alpha) - \alpha.$$

It implies $\alpha - \iota(\alpha)$ is also a 2-torsion element. Since p is coprime to 2, we can conclude that $\alpha = \iota(\alpha)$.

If X is supersingular, then $\text{Km}(X)$ is also supersingular. We have already known that the Brauer group of a supersingular K3 surface is isomorphic to k by [2]. Thus $\text{Br}(X) \cong k$. \square

Remark 2.1.2. In the case A being supersingular, the method of [2] can not be directly applied to show that $\text{Br}(X) = k$ as $H_{\text{dR}}^1(X, \mu_{p^n})$ is not trivial in general for an abelian surface X .

Remark 2.1.3. For abelian surfaces over a non-algebraically closed field or more general ring, we still have the canonical map (2.1.3), but it is not necessarily an isomorphism.

Remark 2.1.4. For a cohomology theory $H^\bullet(-)$ with nice properties e.g. satisfying the blow-up formula, we have a canonical decomposition

$$H^2(\text{Km}(X)) \cong H^2(X) \oplus \pi_* \Sigma,$$

where Σ is the summand in $H^2(\tilde{X})$ generated by the exceptional divisors of $\tilde{\sigma}$.

2.2. A lifting lemma. In [5], Bragg has shown that a twisted K3 surface can be lifted to characteristic 0. Though his method can not be directly applied to twisted abelian surfaces, one can still obtain a lifting result for twisted abelian surfaces via using the Kummer construction. The following result will be frequently used in this paper.

Lemma 2.2.1. *Let $\mathcal{X} \rightarrow X$ be a \mathbb{G}_m -gerbe on an abelian surface X over $k = \bar{k}$. Suppose $\text{char}(k) > 2$ and X has finite height. Then there exists a lifting $\mathfrak{X} \rightarrow \mathcal{X}$ of $\mathcal{X} \rightarrow X$ over some discrete valuation ring W' whose residue field is k such that the specialization map*

$$\text{NS}(\mathcal{X}_{K'}) \rightarrow \text{NS}(X)$$

on Néron–Severi groups is an isomorphism. Here, K' is the fraction field of W' and $\mathcal{X}_{K'}$ is the generic fiber of $\mathcal{X} \rightarrow \text{Spec } W'$.

Proof. The existence of such lifting is ensured by [5, Theorem 7.10], [38, Lemma 3.9] and Proposition 2.1.1. Roughly speaking, let $\mathcal{S} \rightarrow \mathrm{Km}(X)$ be the associated twisted Kummer surface via the isomorphism (2.1.3) in Proposition 2.1.1. Then [5, Theorem 7.10] asserts that there exists a lifting $\mathfrak{S} \rightarrow \mathcal{S}$ of $\mathcal{S} \rightarrow \mathrm{Km}(X)$ such that the specialization map of Néron-Severi groups is an isomorphism

$$\mathrm{NS}(\mathcal{X}_{K'}) \xrightarrow{\sim} \mathrm{NS}(X). \quad (2.2.1)$$

Then [38, Lemma 3.9] says that one can find a lifting \mathcal{X}/W' of X such that $\mathrm{Km}(\mathcal{X}) \cong \mathcal{S}$ over W' . According to Remark 2.1.3, there is a canonical map

$$\Theta : \mathrm{Br}(\mathrm{Km}(\mathcal{X})) \rightarrow \mathrm{Br}(\mathcal{X})$$

as in (2.1.3). Consider the image $\Theta([\mathfrak{S}]) \in \mathrm{Br}(\mathcal{X})$, one can take $\mathfrak{X} \rightarrow \mathcal{X}$ to be the associated twisted abelian surface. Then $\mathfrak{X} \rightarrow \mathcal{X}$ will be a lifting of $\mathcal{X} \rightarrow X$ as the restriction of the Brauer class $[\mathfrak{X}]$ to X is $[\mathcal{X}]$. \square

3. COHOMOLOGICAL REALIZATIONS OF DERIVED ISOGENY

In this section, we briefly recall the action of derived isogenies on the cohomology groups of abelian surfaces and define the prime-to- ℓ derived isogenies. This action has the following two forms

- (1) the motivic realization, which induces rational isomorphisms on the cohomology groups;
- (2) the realization on the integral twisted Mukai lattices.

The story over \mathbb{C} comes back to [66, 33, 31]. Over a general field, we refer [41] for the non-twisted Mukai realization, [40, 6] for the definition of twisted Mukai lattices, and [30, 23] for the motivic realization.

Following the works in [41, 28], we extend the filtered Torelli theorem to twisted abelian surfaces over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. As a corollary, we show that any Fourier–Mukai partner of a twisted abelian surface is isomorphic to a moduli space of stable twisted sheaves on itself or its dual (*cf.* Theorem 3.4.5).

3.1. Motivic realization of derived isogeny on cohomology groups. In [30, 31], Huybrechts shows that (twisted) derived equivalent K3 surfaces over a field k have isomorphic Chow motives, which also holds for general algebraic surfaces over k (as remarked in §2.4 of *loc.cit.*). Moreover, Lie and Vial proved that any twisted derived equivalence induces an isomorphism between the second component of Chows motives by a weight-argument (*cf.* [23, §1.2]). In this part, we record their results for the convenience of the reader. We will focus on abelian surfaces over k as a typical type of examples.

For any abelian surface X over a field k , one may consider idempotent correspondences $\pi_{\mathrm{alg},X}^2$ and $\pi_{\mathrm{tr},X}^2$ in $\mathrm{CH}^2(X \times X)_{\mathbb{Q}}$ defined as

$$\pi_{\mathrm{alg},X}^2 := \sum_{i=1}^{\rho} \frac{1}{\deg(E_i \cdot E_i)} E_i \times E_i, \quad \pi_{\mathrm{tr},X}^2 = \pi_X^2 - \pi_{\mathrm{alg},X}^2,$$

where π_X^2 is the idempotent correspondence given by the Chow–Künneth decomposition (1.2.1) and E_i are non-isotropic divisors generating the Néron–Severi group $\mathrm{NS}(X_{k^s})$ as a orthogonal basis. Consider the decomposition of $\mathfrak{h}^2(X)$:

$$\mathfrak{h}^2(X) = \mathfrak{h}_{\mathrm{alg}}^2(X) \oplus \mathfrak{h}_{\mathrm{tr}}^2(X)$$

given by $\pi_{\mathrm{alg},X}^2$ and $\pi_{\mathrm{tr},X}^2$. It is not hard to see $\mathfrak{h}_{\mathrm{alg}}^2(X)$ is a Tate motive after base change to the separable closure k^s , whose Chow realization is

$$\mathrm{CH}_{\mathbb{Q}}^*(\mathfrak{h}_{\mathrm{alg}}^2(X_{k^s})) \cong \mathrm{NS}(X_{k^s})_{\mathbb{Q}}.$$

According to the main result in [14], any derived equivalence $\mathrm{D}^b(X, \alpha) \xrightarrow{\sim} \mathrm{D}^b(Y, \beta)$ can be uniquely (up to isomorphism) written as a Fourier–Mukai transform with kernel $\mathcal{P} \in \mathrm{D}^b(X \times Y, \alpha^{-1} \boxtimes \beta)$

$$\Phi^{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \xrightarrow{\sim} \mathrm{D}^b(Y, \beta).$$

Consider the cycle class

$$[\Gamma_{\text{tr}}] = v_2(\mathcal{P}) \in \text{CH}^2(\mathcal{X} \times \mathcal{Y})_{\mathbb{Q}} \cong \text{CH}^2(X \times Y)_{\mathbb{Q}},$$

where $v_2(\mathcal{P})$ is the dimension two component of the Mukai vector of \mathcal{P} . It will induce an isomorphism of motives by a weight argument (cf. [23, §§1.2.3])

$$[\Gamma_{\text{tr}}]_2 := \pi_{\text{tr}, Y}^2 \circ [\Gamma_{\text{tr}}] \circ \pi_{\text{tr}, X}^2: \mathfrak{h}_{\text{tr}}^2(X) \xrightarrow{\sim} \mathfrak{h}_{\text{tr}}^2(Y).$$

Since twisted derived equivalent algebraic surfaces have same Picard number (over k), one can choose a invertible correspondence

$$[\Gamma_{\text{alg}}]: \mathfrak{h}_{\text{alg}}^2(X) \xrightarrow{\sim} \mathfrak{h}_{\text{alg}}^2(Y),$$

whose inverse is given by its transpose (see [23, §3.1] for more details). This gives an isomorphism

$$[\Gamma] := [\Gamma_{\text{tr}}]_2 + [\Gamma_{\text{alg}}]: \mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y).$$

Any cohomological realization of such isomorphism clearly preserves the Poincaré pairing by the construction. Therefore, by taking the corresponding cohomological realization, we obtain

Proposition 3.1.1. *Assume $\text{char}(k) = p \neq 2$. Let ℓ be a prime not equal to p . If X and Y are twisted derived equivalent over k , then $[\Gamma]$ will induce a $\text{Gal}(\bar{k}/k)$ -equivariant isometry*

$$\varphi_{\ell}: \text{H}_{\text{ét}}^2(X_{\bar{k}}, \mathbb{Q}_{\ell}) \xrightarrow{\sim} \text{H}_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}). \quad (3.1.1)$$

Suppose k is perfect, it will induce an isometry between F -isocrystals

$$\varphi_K: \text{H}_{\text{crys}}^2(X/K) \xrightarrow{\sim} \text{H}_{\text{crys}}^2(Y/K). \quad (3.1.2)$$

Remark 3.1.2. The weight-argument in [13, §§1.2.3] actually provides an isomorphism

$$\mathfrak{h}(X) \xrightarrow{\sim} \mathfrak{h}(Y),$$

which preserves the even-degree parts

$$\mathfrak{h}^{\text{even}}(-) := \bigoplus_{k=0}^2 \mathfrak{h}^{2k}(-) \cong \bigoplus_{k=0}^2 \bigwedge^{2k} \mathfrak{h}^1(-).$$

The cohomological realizations in Proposition 3.1.1 are not integral in general. We can introduce the prime-to- ℓ derived isogeny via the integral cohomological realizations, which will be used in the rest of the paper.

Definition 3.1.3. Let ℓ be a prime and $\text{char}(k) = p$. When $\ell \neq p$, a derived isogeny $D^b(X) \sim D^b(Y)$ given by

$$\begin{aligned} D^b(X, \alpha) &\xrightarrow{\sim} D^b(X_1, \beta_1) \\ D^b(X_1, \alpha_2) &\xrightarrow{\sim} D^b(X_2, \beta_2) \\ &\vdots \\ D^b(X_n, \alpha_{n+1}) &\xrightarrow{\sim} D^b(Y, \beta_n) \end{aligned}$$

is called *prime-to- ℓ* if each cohomological realization in the zig-zag sequence

$$\varphi_{\ell}^i: \text{H}_{\text{ét}}^2(X_{i-1, \bar{k}}, \mathbb{Q}_{\ell}) \xrightarrow{\sim} \text{H}_{\text{ét}}^2(X_{i, \bar{k}}, \mathbb{Q}_{\ell})$$

is integral, i.e. $\varphi_{\ell}(\text{H}_{\text{ét}}^2(X_{\bar{k}}, \mathbb{Z}_{\ell})) = \text{H}_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_{\ell})$. In the case $\ell = p$, it is called *prime-to- p* if each $\varphi_p^i: \text{H}_{\text{crys}}^2(X_{i-1}/K) \xrightarrow{\sim} \text{H}_{\text{crys}}^2(X_i/K)$ is integral.

3.2. Mukai lattices and \mathbf{B} -fields. At the beginning, we shall remark that we are able to transfer many cohomological statements for twisted K3 surfaces to the case of twisted abelian surfaces via the the Kummer construction thanks to Proposition 2.1.1 and 2.1.4. For this reason, we will omit many technical details which are well-known in the case of K3 surfaces in the following discussions.

If X is a complex abelian surface, the *Mukai lattice* is defined as

$$\tilde{H}(X, \mathbb{Z}) := H^0(X, \mathbb{Z}(-1)) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}(1))$$

with the Mukai pairing

$$\langle (r, c, \chi), (r', c', \chi') \rangle := cc' - r\chi' - r'\chi, \quad (3.2.1)$$

and a pure \mathbb{Z} -Hodge structure of weight 2.

For general algebraically closed field k and an abelian surface X over k , we also have the following notion of Mukai lattices [41, §2].

- Let $\tilde{N}(X)$ be the *extended Néron–Severi lattice* defined as

$$\tilde{N}(X) := \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z},$$

with Mukai pairing

$$\langle (r_1, c_1, \chi_1), (r_2, c_2, \chi_2) \rangle = c_1c_2 - r_1\chi_2 - r_2\chi_1.$$

The Chow realization $\text{CH}_{\mathbb{Q}}^*(-)$ of

$$\mathfrak{h}^0(X) \oplus \mathfrak{h}_{\text{alg}}^2(X) \oplus \mathfrak{h}^4(X)$$

can be identified with $\tilde{N}(X)_{\mathbb{Q}}$.

- if $\text{char}(k) \geq 0$, then the ℓ -adic Mukai lattice is defined on the even degrees of integral ℓ -adic cohomology of X for ℓ coprime to $\text{char}(k)$

$$H_{\text{ét}}^0(X, \mathbb{Z}_{\ell}(-1)) \oplus H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}) \oplus H_{\text{ét}}^4(X, \mathbb{Z}_{\ell}(1)),$$

with Mukai pairing defined in a similar formula as (3.2.1) denoted by $\tilde{H}(X, \mathbb{Z}_{\ell})$; or

- if $\text{char}(k) = p > 0$, then the p -adic Mukai lattice $\tilde{H}(X, W)$ is defined on the even degrees of crystalline cohomology of X with coefficients in $W(k)$

$$H_{\text{crys}}^0(X/W(k))(-1) \oplus H_{\text{crys}}^2(X/W(k)) \oplus H_{\text{crys}}^4(X/W(k))(1),$$

where the twist (i) is given by changing the Frobenius $F \mapsto p^{-i}F$, and the Mukai pairing is given similarly in the formula (3.2.1).

Hodge \mathbf{B} -field. For any $B \in H^2(X, \mathbb{Q})$, we define the *twisted Mukai lattice* as

$$\tilde{H}(X, \mathbb{Z}; B) := \exp(B) \cdot \tilde{H}(X, \mathbb{Z}) \subset \tilde{H}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which is naturally a lattice in $\tilde{H}(X, \mathbb{Z})$, and is equipped with a induced pure Hodge structure of weight 2 from $\tilde{H}(X, \mathbb{Q})$ (cf. [33, Definition 2.3]) i.e.,

$$\tilde{H}^{0,2}(X; B) = \exp(B)\tilde{H}^{0,2}(X).$$

The (extended) twisted Néron–Severi lattice is defined to be $\text{NS}(X; B) := \tilde{H}^{1,1}(X, \mathbb{Z}; B)$.

For such B , we can associate a Brauer class $\alpha_B = \exp(B^{0,2})$ via the exponential sequence

$$H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X) \xrightarrow{\text{exp}} H^2(X, \mathcal{O}_X^*) = \text{Br}(X).$$

Conversely, given $\alpha \in \text{Br}(X)$, one can find a lift B of α in $H^2(X, \mathcal{O}_X)$ because $\text{Br}(X)$ is torsion and $H^3(X, \mathbb{Z})$ is torsion-free. The exponential sequence implies $nB \in H^2(X, \mathbb{Z})$ for the integer n such that $\alpha^n = 1$, and so we have $B \in H^2(X, \mathbb{Q})$. Any such B is called a \mathbf{B} -field lift of α . It is clear that a different choice of such lift B' satisfies $B - B' \in H^2(X, \mathbb{Z})$ by the exponential sequence, and thus there is a Hodge isometry

$$\exp(B - B') : \tilde{H}(X, \mathbb{Z}; B') \xrightarrow{\sim} \tilde{H}(X, \mathbb{Z}; B).$$

This implies that for any Brauer class $\alpha \in \text{Br}(X)$, the twisted Mukai lattice $\tilde{H}(X, \mathbb{Z}; B)$ and the twisted Néron–Severi lattice $\tilde{N}(X; B)$ is independent of the choice of \mathbf{B} -field lift B up to

isometry. Thus for any \mathbb{G}_m -gerbe $\mathcal{X} \rightarrow X$ over a complex abelian surface, we also denote $\tilde{N}(\mathcal{X})$ for the twisted Néron–Severi lattice.

As shown in [33], for any twisted derived equivalence $\Phi^{\mathcal{P}}: D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$, we can associate it with a Hodge isometry

$$\varphi = \varphi_{B, B'}: \tilde{H}(X, \mathbb{Z}; B) \xrightarrow{\sim} \tilde{H}(Y, \mathbb{Z}; B') \quad (3.2.2)$$

for suitable \mathbf{B} -field lifts B, B' of α and β respectively.

ℓ -adic and crystalline \mathbf{B} -field. For the sake of completeness, we will briefly recall the following generalized notions of \mathbf{B} -fields in both ℓ -adic cohomology (cf. [40, §3.2]) and crystalline cohomology (cf. [6, §3]) as an analogue to that in Betti cohomology. We refer [9, §2] for the full consideration of both ℓ -adic and p -adic case, which is for K3 surfaces, but also works for abelian surfaces. The readers who are only interested on our main results may skip this part as we only use these generalized \mathbf{B} -fields in the next subsection and in the supersingular twisted derived Torelli theorem in §§6.6.1.

For a prime $\ell \neq p$ and $n \in \mathbb{N}$, the Kummer sequence of étale sheaves

$$1 \rightarrow \mu_{\ell^n} \rightarrow \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \rightarrow 1, \quad (3.2.3)$$

induces a long exact sequence

$$\cdots \rightarrow \text{Pic}(X) \xrightarrow{!^n} \text{Pic}X \rightarrow H_{\text{ét}}^2(X, \mu_{\ell^n}) \rightarrow \text{Br}(X)[\ell^n] \rightarrow 0.$$

Taking the inverse limit \varprojlim_n , we get a map

$$\pi_{\ell}: H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}(1)) = \varprojlim_n H_{\text{ét}}^2(X, \mu_{\ell^n}) \rightarrow H_{\text{ét}}^2(X, \mu_{\ell^n}) \rightarrow \text{Br}(X)[\ell^n].$$

Lemma 3.2.1. *The map π_{ℓ} is surjective.*

Proof. We have a short exact sequence (cf. [45, Chap.V, Lemma 1.11])

$$0 \rightarrow H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}(1))/\ell^n \rightarrow H_{\text{ét}}^2(X, \mu_{\ell^n}) \rightarrow H_{\text{ét}}^3(X, \mathbb{Z}_{\ell}(1))[\ell^n] \rightarrow 0.$$

As $H_{\text{ét}}^3(X, \mathbb{Z}_{\ell}(1))$ is torsion-free for any abelian surface X , we have an isomorphism

$$H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}(1))/\ell^n \cong H_{\text{ét}}^2(X, \mu_{\ell^n}).$$

Therefore, the reduction morphism $H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}(1)) \rightarrow H_{\text{ét}}^2(X, \mu_{\ell^n})$ can be identified with

$$H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}(1)) \rightarrow H_{\text{ét}}^2(X, \mathbb{Z}_{\ell}(1))/\ell^n,$$

which is surjective. The assertion then follows from it. \square

For any $\alpha \in \text{Br}(X)[\ell^n]$ such that $\ell \neq p$, let $B_{\ell}(\alpha) := \pi_{\ell}^{-1}(\alpha)$, which is non-empty by Lemma 3.2.1.

For Brauer class $\alpha \in \text{Br}(X)[p^n]$, we need the following commutative diagram via the de Rham–Witt theory (cf. [35, I.3.2, II.5.1, Théorème 5.14])

$$\begin{array}{ccccc} 0 & \longrightarrow & H^2(X, \mathbb{Z}_p(1)) & \longrightarrow & H_{\text{crys}}^2(X/W) \xrightarrow{p-F} H_{\text{crys}}^2(X/W) \\ & & \downarrow & & \downarrow p_n := (\otimes W_n) \\ & & H_{\text{fl}}^2(X, \mu_{p^n}) & \xrightarrow{d \log} & H_{\text{crys}}^2(X/W_n) \end{array} \quad (3.2.4)$$

where $H^2(X, \mathbb{Z}_p(1)) := \varprojlim_n H_{\text{fl}}^2(X, \mu_{p^n})$. The $d \log$ map is known to be injective by flat duality (cf. [50, Proposition 1.2]). Since the crystalline cohomology groups of an abelian surface are torsion-free, the mod p^n reduction map p_n is surjective. Consider the canonical surjective map

$$\pi_p: H_{\text{fl}}^2(X, \mu_{p^n}) \rightarrow \text{Br}(X)[p^n],$$

induced by the Kummer sequence. We set

$$B_p(\alpha) := \{b \in H_{\text{crys}}^2(X/W) \mid p_n(b) = d \log(t) \text{ such that } \pi_p(t) = \alpha\}.$$

Following [9, Definition 2.16, 2.17], we can introduce the (mixed) B -fields for twisted abelian surfaces.

Definition 3.2.2. Let ℓ be a prime and let $\alpha \in \text{Br}(X)[\ell^n]$ be a Brauer class of X of order ℓ^n .

- If $\ell \neq p$, an ℓ -adic \mathbf{B} -field lift of α on X is an element $B = \frac{b}{\ell^n} \in \text{H}_{\text{ét}}^2(X, \mathbb{Q}_\ell)$ for some $b \in \text{H}_{\text{ét}}^2(X, \mathbb{Z}_\ell)$ such that $b \in B_\ell(\alpha)$.
- If $\ell = p$, a crystalline \mathbf{B} -field lift of α is an element $B = \frac{b}{p^n} \in \text{H}_{\text{crys}}^2(X/W)[\frac{1}{p}]$ with $b \in \text{H}_{\text{crys}}^2(X/W)$ such that $b \in B_p(\alpha)$.

More generally, for any $\alpha \in \text{Br}(X)$, a mixed \mathbf{B} -field lift of α is a set $B = \{B_\ell\} \cup \{B_p\}$ consisting of a choice of an ℓ -adic \mathbf{B} -field lift B_ℓ of α for each $\ell \neq p$ and a crystalline \mathbf{B} -field lift B_p of α .

Remark 3.2.3. Not all elements in $\text{H}_{\text{crys}}^2(X/W)[\frac{1}{p}]$ are crystalline \mathbf{B} -fields since the map d log is not surjective. From the first row in the diagram (3.2.4), we can see $B \in \text{H}_{\text{crys}}^2(X/W)[\frac{1}{p}]$ is a \mathbf{B} -field lift of some Brauer class if and only if $F(B) = pB$.

For an ℓ -adic or crystalline B -field $B = \frac{b}{m}$, let $\exp(B) = 1 + B + \frac{B^2}{2}$. We define the twisted Mukai lattice as

$$\tilde{\text{H}}(X, B) = \begin{cases} \exp(B)\tilde{\text{H}}(X, \mathbb{Z}_\ell) & \text{if } p \nmid m \\ \exp(B)\tilde{\text{H}}(X, W) & \text{if } m = p^n \end{cases} \quad (3.2.5)$$

under the Mukai pairing (3.2.1). Moreover, for crystalline \mathbf{B} -field B , $\tilde{\text{H}}(X, B)$ is a W -lattice in $\tilde{\text{H}}(X, K)$ stable under the Frobenius action. Sometimes, we denote $\tilde{\text{H}}(\mathcal{X}, \mathbb{Z}_\ell)$ and $\tilde{\text{H}}(\mathcal{X}, W)$ for the twisted Mukai lattices if we want to emphasize the coefficient other than the choice of the \mathbf{B} -field lift.

Now let $\mathcal{X} \rightarrow X$ be a μ_n -gerbe over X whose associated Brauer class is α . The category $\text{Coh}(\mathcal{X})$ of α -twisted coherent sheaves consists of 1-fold \mathcal{X} -twisted coherent sheaves in the sense of Lieblich (cf. [39]), which is proven to be a Grothendieck category. Let $\text{D}^b(\mathcal{X})$ be the bounded derived category of $\text{Coh}(\mathcal{X})$. Consider the Grothendieck group $\text{K}_0(\mathcal{X})$ of $\text{Coh}(\mathcal{X})$. There is a twisted Chern character map

$$\text{ch}_B: \text{K}_0(\mathcal{X}) \rightarrow \tilde{\text{H}}(X, B),$$

see [40, §3.3] and [6, Appendix A3] for ℓ -adic and crystalline cases respectively. The twisted Chern character ch_B factors through the rational extended Néron-Severi lattice $\tilde{N}(X)_\mathbb{Q}$:

$$\begin{array}{ccc} \text{K}_0(\mathcal{X}) & \xrightarrow{\text{ch}_B} & \tilde{\text{H}}(X, B) \\ & \searrow \text{ch}_{\mathcal{X}} & \nearrow \exp(B) \text{cl}_H \\ & & \tilde{N}(X)_\mathbb{Q}, \end{array}$$

where cl_H is the cycle class map. The image of $\text{K}_0(\mathcal{X})$ in $\tilde{N}(X)_\mathbb{Q}$ under ch_B is denoted by $\tilde{N}(\mathcal{X})$. For any \mathcal{X} -twisted sheaf \mathcal{E} on X , the Mukai vector $v_B(\mathcal{E})$ is defined to be

$$\text{ch}_B([\mathcal{E}])\sqrt{\text{Td}(X)} \in \tilde{\text{H}}(X, B).$$

Since the Todd class $\text{Td}(X)$ is trivial when X is an abelian surface, $v_B(\mathcal{E}) = \text{ch}_B([\mathcal{E}]) \in \tilde{\text{H}}(X, B)$.

For any Fourier–Mukai transform $\Phi^{\mathcal{P}}: \text{D}^b(\mathcal{X}) \rightarrow \text{D}^b(\mathcal{Y})$, [9, Theorem 3.6] shows that there is an isometry of Mukai lattices for suitable (mixed) \mathbf{B} -field lifts B and B'

$$\varphi_{B, B'}: \tilde{\text{H}}(X, B) \rightarrow \tilde{\text{H}}(Y, B'). \quad (3.2.6)$$

3.3. A filtered Torelli Theorem. In [41, 42], Lieblich and Olsson introduce the notion of filtered derived equivalence and show that filtered derived equivalent K3 surfaces are isomorphic. In this part, we will give an analogue for (twisted) abelian surfaces, whose proof is much more simple than the K3 surface case as the bounded derived category of a (twisted) abelian surface is a generic K3 category in the sense of [32].

The rational numerical Chow ring $\mathrm{CH}_{\mathrm{num}}^*(X)_{\mathbb{Q}}$ is equipped with a codimension filtration

$$\mathrm{Fil}^i \mathrm{CH}_{\mathrm{num}}^*(X)_{\mathbb{Q}} := \bigoplus_{i \geq k} \mathrm{CH}_{\mathrm{num}}^k(X)_{\mathbb{Q}}.$$

As X is a surface, we have a natural identification $\tilde{N}(X)_{\mathbb{Q}} \cong \mathrm{CH}_{\mathrm{num}}^*(X)_{\mathbb{Q}}$, which gives a filtration of the rational extended Néron-Severi lattice. Let $\Phi^{\mathcal{P}}$ be a Fourier-Mukai transform with respect to $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. The equivalence $\Phi^{\mathcal{P}}$ is called *filtered* if the induced numerical Chow realization $\Phi_{\mathrm{CH}}^{\mathcal{P}}$ preserves the codimension filtration. It is not hard to see that $\Phi^{\mathcal{P}}$ is filtered if and only if it sends the Mukai vector $(0, 0, 1)$ to $(0, 0, 1)$. A filtered twisted Fourier-Mukai transform is defined in a same way since the twisted Chern character $\mathrm{ch}_{\mathcal{X}}$ maps onto $\tilde{N}(\mathcal{X}) \subset \tilde{N}(X)_{\mathbb{Q}}$.

At the cohomological level, the codimension filtration on $\tilde{H}(X)[\frac{1}{\ell}]$ (the prime ℓ depends on the choice of ℓ -adic or crystalline twisted Mukai lattice) is given by $F^i = \bigoplus_{r \geq i} \mathrm{H}^{2r}(X)[\frac{1}{\ell}]$. Let B be a \mathbf{B} -field lift of $[\mathcal{X}]$. The filtration on $\tilde{H}(X, B)$ is defined by

$$F^i \tilde{H}(X, B) = \tilde{H}(X, B) \cap F^i \tilde{H}(X)[\frac{1}{\ell}].$$

A direct computation shows that the graded pieces of F^{\bullet} are

$$\begin{aligned} \mathrm{Gr}_F^0 \tilde{H}(X, B) &= \left\{ (r, rB, \frac{rB^2}{2}) \mid r \in \mathrm{H}^0(X) \right\}, \\ \mathrm{Gr}_F^1 \tilde{H}(X, B) &= \{ (0, c, c \cdot B) \mid c \in \mathrm{H}^2(X) \} \cong \mathrm{H}^2(X), \\ \mathrm{Gr}_F^2 \tilde{H}(X, B) &= \{ (0, 0, s) \mid s \in \mathrm{H}^4(X) \} \cong \mathrm{H}^4(X)(1). \end{aligned} \tag{3.3.1}$$

Lemma 3.3.1. *A twisted Fourier-Mukai transform $\Phi^{\mathcal{P}} : \mathrm{D}^b(\mathcal{X}) \rightarrow \mathrm{D}^b(\mathcal{Y})$ is filtered if and only if its cohomological realization is filtered for certain \mathbf{B} -field lifts.*

Proof. It is clear that being filtered implies being cohomologically filtered. This is because the map $\exp(B) \cdot \mathrm{cl}_H : \tilde{N}(X, \mathbb{Q}) \rightarrow \tilde{H}(X, B)$ preserves the filtrations for any \mathbf{B} -field lift B of $[\mathcal{X}]$.

For the converse, just notice that $\Phi^{\mathcal{P}}$ is filtered if and only if the induced map $\Phi_{\mathrm{CH}}^{\mathcal{P}}$ takes the vector $(0, 0, 1)$ to $(0, 0, 1)$. As $\Phi^{\mathcal{P}}$ is cohomologically filtered for B , we can see the cohomological realization of $\Phi^{\mathcal{P}}$ preserves the graded piece Gr_F^2 in (3.3.1). This implies that $\Phi_{\mathrm{CH}}^{\mathcal{P}}$ takes $(0, 0, 1)$ to $(0, 0, 1)$. \square

Proposition 3.3.2 (filtered Torelli theorem for twisted abelian surfaces). *Suppose $k = \bar{k}$. Let $\mathcal{X} \rightarrow X$ and $\mathcal{Y} \rightarrow Y$ be μ_n -gerbes on abelian surfaces. Then following statements are equivalent*

- (1) *There is an isomorphism between associated \mathbb{G}_m -gerbes $\mathcal{X}_{\mathbb{G}_m}$ and $\mathcal{Y}_{\mathbb{G}_m}$.*
- (2) *There is a filtered Fourier-Mukai transform $\Phi^{\mathcal{P}}$ from \mathcal{X} to \mathcal{Y} .*

Proof. For untwisted case, i.e. $\mathcal{X} = X$ and $\mathcal{Y} = Y$, this is exactly [28, Proposition 3.1]. Here we extend it to the twisted case. As one direction is obvious, it suffices to show that (2) can imply (1). Set

$$\mathcal{P}_x := \Phi^{\mathcal{P}}(k(x)) = \mathcal{P}|_{\{x\} \times Y},$$

the image of the skyscraper sheaf $k(x)$ for a closed point $x \in X$. Since $\mathrm{Coh}(\mathcal{Y})$ admits no spherical objects (cf. [32, §§3.2]), $\mathrm{D}^b(\mathcal{Y})$ are generic K3-categories and the semi-rigid objects in $\mathrm{D}^b(\mathcal{Y})$ are in $\mathrm{Coh}(\mathcal{Y})$ up to shift of degree. We can see there is an integer m such that $H^i(\mathcal{P}_x) = 0$ for all $i \neq m$ and closed points x (cf. [32, Proposition 3.18]). Therefore, there is a $\mathcal{X}^{(-1)} \times \mathcal{Y}$ -twisted sheaf $\mathcal{E} \in \mathrm{Coh}(\mathcal{X}^{(-1)} \times \mathcal{Y})$ such that $\mathcal{P} \cong \mathcal{E}[m]$. Since $\Phi_{\mathcal{X} \rightarrow \mathcal{Y}}^{\mathcal{P}}$ sends $(0, 0, 1)$ to $(0, 0, 1)$, \mathcal{E}_x is just a skyscraper sheaf on $\{x\} \times Y$. Then one can proceed the arguments as in [14, Corollary 5.3] or [29, Corollary 5.22, 5.23] to show that there is an isomorphism $f : X \rightarrow Y$ such that $f^*([\mathcal{Y}_{\mathbb{G}_m}]) = [\mathcal{X}_{\mathbb{G}_m}]$. \square

3.4. Twisted FM partners via moduli space of twisted sheaves. Keep the notations as before, we denote by $\mathcal{M}_H(\mathcal{X}, v)$ (or $\mathcal{M}_H^\alpha(X, v)$) the moduli stack of H -semistable \mathcal{X} -twisted sheaves with Mukai vector $v \in \tilde{N}(\mathcal{X})$, where H is a v -generic ample divisor on X and $\alpha = [\mathcal{X}]$ the associated Brauer class of X (cf. [39] or [66]). To characterize the Fourier–Mukai partners of twisted abelian surfaces via the moduli space of twisted sheaves, we first need the following criterion on non-emptiness of moduli space of (twisted) sheaves on an abelian surface X in positive characteristic. In the rest of this section, we will always assume that $k = \bar{k}$ and $\text{char}(k) \neq 2$.

Proposition 3.4.1 (Minamide–Yanagida–Yoshioka, Bragg–Lieblich). *Let n be a positive integer. Assume that either $p \nmid n$ or X is supersingular and $n = p$. Let $\mathcal{X} \rightarrow X$ be a μ_n -gerbe on X . Let $v = (r, \ell, s) \in \tilde{N}(\mathcal{X})$ be a primitive Mukai vector such that $v^2 = 0$. Fix a v -generic ample divisor H . If one of the following holds (called positive):*

- (1) $r > 0$.
- (2) $r = 0$ and ℓ is effective.
- (3) $r = \ell = 0$ and $s > 0$.

then the coarse moduli space $M_H(\mathcal{X}, v) \neq \emptyset$ and the moduli stack $\mathcal{M}_H(\mathcal{X}, v)$ is a \mathbb{G}_m -gerbe on $M_H(\mathcal{X}, v)$. Moreover, its coarse moduli space $M_H(\mathcal{X}, v)$ is an abelian surface.

Proof. If $\mathcal{X} \rightarrow X$ is a μ_n -gerbe such that $p \nmid n$, then the statements are proven in [46, Proposition A.2.1] which is based on a statement of lifting a Brauer classes on X to characteristic 0 which requires the condition $p \nmid n$ (see Lemma A.2.3 in loc.cit.).

If X is supersingular and $\mathcal{X} \rightarrow X$ is a μ_p -gerbe, then the assertion will follow from a same argument in [7, Proposition 4.1.20], as we will see in §6.6.2 that the twisor space of a supersingular abelian surface can be constructed. \square

Remark 3.4.2. Actually, one can obtain the non-emptiness of $\mathcal{M}_H(\mathcal{X}, v)$ for a μ_n -gerbe $\mathcal{X} \rightarrow X$ over an abelian surface of finite height with $p \mid n$ by following [46, Proposition A.2.1] together with the lifting result 2.2.1.

Remark 3.4.3. In the case $\mathcal{X} \rightarrow X$ is a essentially-trivial μ_p -gerbe over a supersingular abelian surface X , this can be proved by a standard lifting argument (see also [22, Proposition 6.9]). When $\mathcal{X} \rightarrow X$ is non-trivial, Bragg–Lieblich’s approach is to take the universal family of μ_p -gerbes

$$f: \mathfrak{X} \rightarrow \mathbb{A}^1$$

on the connected component $\mathbb{A}^1 \subset \mathbb{R}^2 \pi_* \mu_p$ which contains \mathcal{X} (cf. Corollary 6.6.6). The fibers of f contain $\mathcal{X} \rightarrow X$ and the trivial μ_p -gerbe over X . By taking the relative moduli space of twisted sheaves (with suitable v -generic polarization) on $\mathfrak{X} \rightarrow \mathbb{A}^1$, one can see the non-emptiness of $M_H(\mathcal{X}, v)$ from the case of essentially trivial gerbes.

Now, we are going to define the twisted Poincaré bundle for a gerbe on a given abelian surface. Let $\mathcal{X} \rightarrow X$ be a μ_n -gerbe on X such that $p \nmid n$. As an element in $H_{\text{ét}}^2(X, \mu_n)$, we can (uniquely) associate \mathcal{X} with a symmetric morphism

$$\varphi_n: X[n] \rightarrow \widehat{X}[n]$$

by the Weil pairing (cf. [59, Lemma 16.22]). Dually, we have $\varphi_n^t: \widehat{X}[n] \rightarrow X[n]$, which corresponds to a μ_n -gerbe on \widehat{X} , denoted by $\widehat{\mathcal{X}}$. We can take a separable isogeny $f: Y \rightarrow X$ such that $\widehat{f}[n] \circ \varphi_n \circ f[n] = 0$. This implies $f^* \mathcal{X} = 0 \in H_{\text{ét}}^2(Y, \mu_n)$. Then there is also a separable isogeny $f^t: \widehat{Y} \rightarrow \widehat{X}$ given by the Cartier dual $\ker(f)^D \subset \widehat{Y}$, which satisfies $f^{t*} \widehat{\mathcal{X}} = 0$. Let \mathcal{P}_0 be the Poincaré bundle on $Y \times \widehat{Y}$. Consider

$$f \times f^t: Y \times \widehat{Y} = V \rightarrow X \times \widehat{X}$$

as a finite étale covering which trivializes the μ_n -gerbe $\mathcal{X} \times \widehat{\mathcal{X}}$, we will get a $\mathcal{X} \times \widehat{\mathcal{X}}$ -twisted sheaf $\mathcal{P}_{\mathcal{X}}$ on $X \times \widehat{X}$ by the étale descent. We have the following commutative diagram

$$\begin{array}{ccccc}
 & & V & & \\
 & \swarrow p_Y & \downarrow f \times f^t & \searrow q_Y & \\
 Y & & X \times \widehat{X} & & \widehat{Y} \\
 \downarrow f & \swarrow p_X & & \searrow q_X & \downarrow f^t \\
 X & & & & \widehat{X}
 \end{array}$$

Proposition 3.4.4. *The Fourier–Mukai functor $\Phi^{\mathcal{P}_X} : D^b(\mathcal{X}^{(-1)}) \rightarrow D^b(\widehat{\mathcal{X}})$ is a derived equivalence.*

Proof. This statement can be checked étale locally on \widehat{X} . Then this follows from the Bridgeland’s criterion (Theorem 2.3 and Theorem 3.3 in [11]) as in [29, Proposition 9.19], since \mathcal{P}_X is étale locally the Poincaré bundle: For any skyscraper sheaf $k(x)$ on X , which is naturally a $\mathcal{X}^{(-1)}$ -twisted sheaf, we have the following yoga

$$\begin{aligned}
 \Phi^{\mathcal{P}_X}(k(x))|_{\widehat{Y}} &= (f^t)^* q_{X*}(\mathcal{P}_{\mathcal{X}} \otimes p_X^* k(x)) \\
 &= q_{Y*}(f \times f^t)^*(\mathcal{P}_{\mathcal{X}} \otimes p_X^* k(x)) \\
 &\cong q_{Y*}(\mathcal{P}_0 \otimes p_Y^* f^* k(x)) \\
 &\cong \bigoplus_{y \in f^{-1}(x)} q_{Y*}(\mathcal{P}_0 \otimes p_Y^*(k(y))) = \bigoplus_{y \in f^{-1}(x)} \mathcal{P}_{0,y}.
 \end{aligned}$$

where $\mathcal{P}_{0,y}$ is the line bundle on $\{x\} \times \widehat{Y}$ corresponding to $y \in Y \cong \text{Pic}^0(\widehat{Y})$. \square

The following is an extension of [28, Theorem 1.2].

Theorem 3.4.5. *With the same assumptions as in Proposition 3.4.1. Let $\mathcal{X} \rightarrow X$ be μ_n -gerbe on an abelian surface X such that $p \nmid n$. Then the associated \mathbb{G}_m -gerbe of any Fourier–Mukai partner of \mathcal{X} is isomorphic to a \mathbb{G}_m -gerbe on the moduli space of \mathcal{Y} -twisted sheaves $M_H(\mathcal{Y}, v)$ with \mathcal{Y} being \mathcal{X} or $\widehat{\mathcal{X}}$.*

Proof. Let \mathcal{M} be a Fourier–Mukai partner of \mathcal{X} . Let $\Phi_{\mathcal{M} \rightarrow \mathcal{X}}^{\mathcal{P}}$ be the Fourier–Mukai transform. Let v be the image of $(0, 0, 1)$ under $\Phi_{\mathcal{M} \rightarrow \mathcal{X}}^{\mathcal{P}}$. We can assume v satisfying one of the conditions in Proposition 3.4.1 by changing \mathcal{X} to $\widehat{\mathcal{X}}$ if necessary. It is proved that the moduli stack $\mathcal{M}_H(\mathcal{X}, v)$ is a \mathbb{G}_m -gerbe on $M_H(\mathcal{X}, v)$ in Proposition 3.4.1. Then there is a Fourier–Mukai transform

$$\Phi^{\mathcal{P}} : D^b(\mathcal{M}_H(\mathcal{X}, v)^{(-1)}) \rightarrow D^b(\mathcal{X}^{(1)}) \quad (3.4.1)$$

induced by the tautological sheaf \mathcal{P} on $\mathcal{M}_H(\mathcal{X}, v) \times \mathcal{X}$, whose cohomological realization maps the Mukai vector $(0, 0, 1)$ to v . Combining it with the derived equivalence

$$\Phi : D^b(\mathcal{X}) \rightarrow D^b(\mathcal{M}),$$

we will obtain a filtered derived equivalence from $\mathcal{M}_H(\mathcal{X}, v)^{(-1)}$ to $\mathcal{M}^{(1)}$. This induces an isomorphism from $\mathcal{M}_H(\mathcal{X}, v)^{(-1)}$ to $\mathcal{M}_{\mathbb{G}_m}^{(1)}$ by Theorem 3.3.2. \square

4. SHIODA’S TORELLI THEOREM FOR ABELIAN SURFACES

In [58], Shioda noticed that there is a way to extract the information of the 1st-cohomology of a complex abelian surface from its 2nd-cohomology, called Shioda’s trick. This established a global Torelli theorem for complex abelian surfaces via the 2nd-cohomology, which is also a key step in Pjateckii–Šapiro–Šafarevič’s proof of the Torelli theorem for K3 surfaces (*cf.* [53, Lemma 4, Theorem 1]).

The aim of this section is to generalize Shioda’s method to all fields and establish an isogeny theorem for abelian surfaces via the 2nd-cohomology. We will deal with Shioda’s trick for Betti cohomology, étale cohomology and crystalline cohomology separately.

4.1. Recap of Shioda's trick for Hodge isometry. We first recall Shioda's construction. Suppose X is a complex abelian surface. Its singular cohomology ring $H^\bullet(X, \mathbb{Z})$ is canonically isomorphic to the exterior algebra $\wedge^\bullet H^1(X, \mathbb{Z})$. Let V be a free \mathbb{Z} -module of rank 4. We denote by Λ the lattice $(\wedge^2 V, q)$ where $q : \wedge^2 V \times \wedge^2 V \rightarrow \mathbb{Z}$ is the wedge product. After choosing a \mathbb{Z} -basis $\{v_i\}_{1 \leq i \leq 4}$ for $H^1(X, \mathbb{Z})$, we have an isometry of \mathbb{Z} -lattice $\Lambda \xrightarrow{\sim} H^2(X, \mathbb{Z})$. The set of vectors

$$\{v_{ij} := v_i \wedge v_j\}_{0 \leq i < j \leq 4}$$

clearly forms a basis of $H^2(X, \mathbb{Z})$, which will be called an *admissible basis* of A for its second singular cohomology. For another complex abelian surface Y , a Hodge isometry

$$\psi : H^2(Y, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$$

will be called *admissible* if $\det(\psi) = 1$, with respect to some admissible bases on X and Y . It is clear that the admissibility of a morphism is independent of the choice of admissible bases.

In terms of admissible bases, we can view ψ as an element in $\mathrm{SO}(\Lambda)$. On the other hand, we have the following exact sequence of groups

$$1 \rightarrow \{\pm 1\} \rightarrow \mathrm{SL}_4(\mathbb{Z}) \xrightarrow{\wedge^2} \mathrm{SO}(\Lambda) \quad (4.1.1)$$

Shioda observed that the image of $\mathrm{SL}_4(\mathbb{Z})$ in $\mathrm{SO}(\Lambda)$ is a subgroup of index two and does not contain $-\mathrm{id}_\Lambda$. From this, he proved the following (cf. [58, Theorem 1])

Theorem 4.1.1 (Shioda). *For any admissible integral Hodge isometry ψ , there is an isomorphism of integral Hodge structures*

$$\varphi : H^1(Y, \mathbb{Z}) \xrightarrow{\sim} H^1(X, \mathbb{Z})$$

such that $\wedge^2(\varphi) = \psi$ or $-\psi$.

This is what we call ‘‘Shioda's trick’’. As we can assume a Hodge isometry being admissible after possibly taking the dual abelian variety for one of them, we can obtain the Torelli theorem for complex abelian surfaces by using the weight two Hodge structures, i.e., X is isomorphic to Y or its dual \hat{Y} if and only if there is an integral Hodge isometry $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ (cf. [58, Theorem 1]).

4.2. Admissible basis. In order to extend Shioda's work to arbitrary fields, we need to define admissibility for various cohomology theories (e.g. étale cohomology and crystalline cohomology).

Let k be a perfect field with $\mathrm{char}(k) = 0$ or $p \geq 2$. Suppose X is an abelian surface over k and $\ell \nmid p$ is a prime. For simplicity of notations, we will denote $H^\bullet(-)_R$ for one of the following cohomology theories:

- (1) if $k \hookrightarrow \mathbb{C}$ and $R = \mathbb{Z}$ or any number field E , then $H^\bullet(X)_R = H^\bullet(X(\mathbb{C}), R)$ the singular cohomology.
- (2) if $R = \mathbb{Z}_\ell$ or \mathbb{Q}_ℓ , then $H^\bullet(X)_R = H_{\text{ét}}^\bullet(X_{\bar{k}}, R)$, the ℓ -adic étale cohomology.
- (3) if $\mathrm{char}(k) = p > 0$ and $R = W$ or K , then $H^\bullet(X)_R = H_{\text{crys}}^\bullet(X_{k^{\text{perf}}}/W)$ or $H_{\text{crys}}^\bullet(X_{k^{\text{perf}}}/W) \otimes K$, the crystalline cohomology.

There is an isomorphism between the cohomology ring $H^\bullet(X)_R$ and the exterior algebra $\wedge^\bullet H^1(X)_R$. We denote by $\mathrm{tr}_X : H^4(X)_R \xrightarrow{\sim} R$ the corresponding trace map. Then the Poincaré pairing $\langle -, - \rangle$ on $H^2(X)_R$ can be realized as

$$\langle \alpha, \beta \rangle = \mathrm{tr}_X(\alpha \wedge \beta).$$

Analogous to §4.1, a R -basis $\{v_i\}$ of $H^1(X)_R$ will be called a *d-admissible basis* if it satisfies

$$\mathrm{tr}_X(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = d$$

for some $d \in R^*$. When $d = 1$, it will be called an *admissible basis*. For any d -admissible (resp. admissible) basis $\{v_i\}$, the associated R -basis $\{v_{ij} := v_i \wedge v_j\}_{i < j}$ of $H^2(X)_R$ will also be called d -admissible (resp. admissible).

Example 4.2.1. Let $\{v_1, v_2, v_3, v_4\}$ be a R -linear basis of $H^1(X)_R$. Suppose

$$\mathrm{tr}_X(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = t \in R^*.$$

For any $d \in R^*$, there is a natural d -admissible R -linear basis $\{\frac{d}{t}v_1, v_2, v_3, v_4\}$

Definition 4.2.2. Let X and Y be abelian surfaces over k .

- a R -linear isomorphism $\psi: H^1(X)_R \rightarrow H^1(Y)_R$ is d -admissible if it takes an admissible basis to a d -admissible basis.
- a R -linear isomorphism $\varphi: H^2(X)_R \rightarrow H^2(Y)_R$ is d -admissible if

$$\mathrm{tr}_Y \circ \wedge^2(\varphi) = d \mathrm{tr}_X$$

for some $d \in R^*$, or equivalently, it sends an admissible basis to a d -admissible basis. When $d = 1$, it will also be called admissible.

The set of d -admissible isomorphisms will be denoted by $\mathrm{Iso}^{\mathrm{ad},(d)}(H^1(X)_R, H^1(Y)_R)$ and $\mathrm{Iso}^{\mathrm{ad},(d)}(H^2(X)_R, H^2(Y)_R)$ respectively.

For any isomorphism $\varphi: H^2(X)_R \xrightarrow{\sim} H^2(Y)_R$, let $\det(\varphi)$ be the determinant of the matrix with respect to some admissible bases. It is not hard to see $\det(\varphi)$ is independent of the choice of admissible bases, and φ is admissible if and only if $\det(\varphi) = 1$.

Example 4.2.3. For the dual abelian surface \widehat{X} , the dual basis $\{v_i^*\}$ with respect to the Poincaré pairing naturally forms an admissible basis, under the identification $H^1(X)_R^\vee \cong H^1(\widehat{X})_R$. Let

$$\psi_{\mathcal{P}}: H^2(X)_R \rightarrow H^2(\widehat{X})_R$$

be the isomorphism induced by the Poincaré bundle \mathcal{P} on $X \times \widehat{X}$. A direct computation (see e.g. [29, Lemma 9.3]) shows that $\psi_{\mathcal{P}}$ is nothing but

$$-D: H^2(X)_R \xrightarrow{\sim} H^2(X)_R^\vee \cong H^2(\widehat{X})_R,$$

where D is the Poincaré duality. For an admissible basis $\{v_i\}$ of X , its R -linear dual $\{v_i^*\}$ with respect to Poincaré pairing forms an admissible basis of \widehat{X} . By our construction, we can see

$$D(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) = (v_{34}^*, -v_{24}^*, v_{23}^*, v_{14}^*, -v_{13}^*, v_{12}^*),$$

which implies that D is of determinant -1 under these admissible bases. Thus the determinant of $\psi_{\mathcal{P}}$ is not admissible.

Example 4.2.4. Let $f: X \rightarrow Y$ be an isogeny of degree d for some $d \in \mathbb{Z}_{\geq 0}$ between two abelian surfaces. If d is coprime to ℓ , then it will induce an isomorphism

$$f^*: H^2(Y)_{\mathbb{Z}_\ell} \xrightarrow{\sim} H^2(X)_{\mathbb{Z}_\ell},$$

which is d -admissible. If $d = n^4$, then $\frac{1}{n}f^*$ will be an admissible \mathbb{Z}_ℓ -integral isometry with respect to the Poincaré pairing.

If $\ell \neq 2$, then d or $-d$ is a square in \mathbb{Z}_ℓ . Thus there is some $\xi \in \mathbb{Z}_\ell^*$ such that $\pm d = \xi^4$. Therefore, we can always find an admissible \mathbb{Z}_ℓ -integral isomorphism $\frac{1}{\xi}f^*: H^2(Y)_{\mathbb{Z}_\ell} \rightarrow H^2(X)_{\mathbb{Z}_\ell}$ by possibly changing Y to \widehat{Y} .

Example 4.2.5. Suppose X is an abelian surface over a perfect field k with $\mathrm{char}(k) = p > 0$. Then F -crystal $H^1(X)_W$ together with the trace map

$$\mathrm{tr}_X: H^4(X)_W \xrightarrow{\sim} W$$

form an abelian crystal, in the sense of [50, §6]. We can see $H^1(X)_W \cong H^1(Y)_W$ as abelian crystals if and only if there is an admissible isomorphism $H^1(X)_W \xrightarrow{\sim} H^1(Y)_W$.

4.3. More on admissible basis of F -crystals. In contrast to ℓ -adic étale cohomology, the semilinear structure on crystalline cohomology from its Frobenius is more tricky to work with. Therefore, it seems necessary for us to spend more words on the interaction of Frobenius with admissible bases.

We have the following Frobenius pull-back diagram:

$$\begin{array}{ccccc}
 X & & & & \\
 \searrow^{F_X^{(1)}} & & \xrightarrow{F_X} & & \\
 & X^{(1)} & \longrightarrow & X & \\
 & \downarrow & & \downarrow & \\
 & \text{Spec}(k) & \xrightarrow{\sigma} & \text{Spec}(k) &
 \end{array}$$

Via the natural identification $H_{\text{crys}}^1(X^{(1)}/W) \cong H_{\text{crys}}^1(X/W) \otimes_{\sigma} W$, the σ -linearization of Frobenius action on $H_{\text{crys}}^1(X/W)$ can be viewed as the injective W -linear map

$$F^{(1)} := \left(F_X^{(1)}\right)^* : H_{\text{crys}}^1(X^{(1)}/W) \hookrightarrow H_{\text{crys}}^1(X/W).$$

There is a decomposition $H_{\text{crys}}^1(X/W) = H_0(X) \oplus H_1(X)$ such that

$$F^{(1)} \left(H_{\text{crys}}^1(X^{(1)}/W) \right) \cong H_0(X) \oplus pH_1(X), \quad (4.3.1)$$

and $\text{rank}_W H_i = 2$ for $i = 0, 1$, which is related to the Hodge decomposition of the de Rham cohomology of X/k by Mazur's theorem; see [4, §8, Theorem 8.26].

The Frobenius map can be expressed in terms of admissible basis. We can choose an admissible basis $\{v_i\}$ of $H_{\text{crys}}^1(X/W)$ such that

$$v_1, v_2 \in H_0(X) \quad \text{and} \quad v_3, v_4 \in H_1(X).$$

Then $\{p^{\alpha_i} v_i\} := \{v_1, v_2, pv_3, pv_4\}$ forms an admissible basis of $H_{\text{crys}}^1(X^{(1)}/W)$ under the identification (4.3.1), since $\text{tr}_p \circ \wedge^4 F^{(1)} = p^2 \sigma_W \circ \text{tr}_p$. In term of these basis, the Frobenius map can be written as

$$F^{(1)}(p^{\alpha_i} v_i) = \sum_j c_{ij} p^{\alpha_j} v_j,$$

where $C_X = (c_{ij})$ forms an invertible 4×4 -matrix with coefficients in W .

Suppose Y is another abelian surface over k and $\rho: H_{\text{crys}}^1(X/W) \rightarrow H_{\text{crys}}^1(Y/W)$ is an admissible map. Denote $\rho^{(1)}$ for the induced map $\rho \otimes_{\sigma} W: H_{\text{crys}}^1(X^{(1)}/W) \rightarrow H_{\text{crys}}^1(Y^{(1)}/W)$. The following lemma is clear.

Lemma 4.3.1. *The map ρ is a morphism between F -crystals if and only if $C_Y^{-1} \cdot \rho^{(1)} \cdot C_X = \rho$, where “ \cdot ” denotes by the action of matrix with respect to the chosen admissible bases.*

4.4. Generalized Shioda's trick. Let us review some basic properties of the special orthogonal group scheme over an integral domain. Let Λ be an even \mathbb{Z} -lattice of rank $2n$. Then we can associate it with a vector bundle $\underline{\Lambda}$ on $\text{Spec}(\mathbb{Z})$ with constant rank $2n$ equipped with a quadratic form q over $\text{Spec}(\mathbb{Z})$ obtained from Λ . Then the functor

$$A \mapsto \{g \in \text{GL}(\Lambda_A) \mid q_A(g \cdot x) = q_A(x) \text{ for all } x \in \Lambda_A\}$$

represents a \mathbb{Z} -subscheme of $\text{GL}(\Lambda)$, denoted by $\text{O}(\Lambda)$. There is a homomorphism between \mathbb{Z} -group schemes

$$D_{\Lambda}: \text{O}(\Lambda) \rightarrow \underline{\mathbb{Z}/2\mathbb{Z}},$$

which is called the Dickson morphism. It is surjective as Λ is even, and its formation commutes with any base change. The *special orthogonal group scheme* over \mathbb{Z} with respect to Λ is defined to be the kernel of D_{Λ} , which is denoted by $\text{SO}(\Lambda)$. Moreover, we have

$$\text{SO}(\Lambda)_{\mathbb{Z}[\frac{1}{2}]} \cong \ker(\det: \text{O}(\Lambda) \rightarrow \mathbb{G}_m)_{\mathbb{Z}[\frac{1}{2}]}.$$

It is well-known that $\mathrm{SO}(\Lambda) \rightarrow \mathrm{Spec}(\mathbb{Z})$ is smooth of relative dimension $\frac{n(n-1)}{2}$ and with connected fibers; see [16, Theorem C.2.11] for instance. Moreover, it is well-known that the special orthogonal group scheme admits a universal covering (i.e., a simply connected central isogeny)

$$\mathrm{Spin}(\Lambda) \rightarrow \mathrm{SO}(\Lambda).$$

See Appendix C.4 in loc.cit. for the construction. For any ℓ , the special orthogonal group scheme

$$\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}) \cong \mathrm{SO}(\Lambda)_{\mathbb{Z}_\ell}$$

is smooth over \mathbb{Z}_ℓ with connected fibers, which implies its generic fiber $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell})$ is connected. Thus $\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is clearly connected as a group scheme over \mathbb{Z}_ℓ as $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell}) \subset \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is dense.

Let V be free \mathbb{Z} -module of rank 4 and $\Lambda = \wedge^2 V$ with the natural pairing. Let R be a ring of coefficients as listed in §§4.2. Then we have

Lemma 4.4.1. *There is an exact sequence of smooth R -group schemes*

$$1 \rightarrow \mu_{2,R} \rightarrow \mathrm{SL}(V)_R \xrightarrow{\wedge^2(-)_R} \mathrm{SO}(\Lambda)_R \rightarrow 1.$$

(as fppf-sheaves if $\frac{1}{2} \notin R$.) Moreover, there is an exact sequence

$$1 \rightarrow \{\pm \mathrm{id}_4\} \rightarrow \mathrm{SL}(V)(R) \xrightarrow{\wedge^2(-)_R} \mathrm{SO}(\Lambda)(R) \xrightarrow{\mathrm{SN}} R^*/(R^*)^2, \quad (4.4.1)$$

where SN is the map of spinor norm (see [3, §3.3] for the definition).

Proof. For the first statement, it suffices to assume $R = \mathrm{Spec}(\bar{k})$ for an algebraically closed field \bar{k} , where it is clear from a computation.

Note that we have an exact sequence on rational points (cf. [25, Proposition 3.2.2])

$$1 \rightarrow \mu_2(R) \rightarrow \mathrm{SL}(V)(R) \rightarrow \mathrm{SO}(\Lambda)(R) \rightarrow \mathrm{H}^1(\mathrm{Spec}(R), \mu_2).$$

From the Kummer sequence for μ_2 , we can see

$$\mathrm{H}^1(\mathrm{Spec}(R), \mu_2) \cong \mathrm{H}_{\text{ét}}^1(\mathrm{Spec}(R), \mu_2) \cong R^*/(R^*)^2$$

as $\mathrm{Pic}(R)[2] = 0$.

For the last statement, it is sufficient to see that there is an isomorphism of R -group schemes $\mathrm{SL}(V)_R \xrightarrow{\sim} \mathrm{Spin}(\Lambda)_R$ such that the following diagram commutes

$$\begin{array}{ccccc} & & \mathrm{Spin}(\Lambda)(R) & & \\ & \nearrow \sim & \downarrow & & \\ \mathrm{SL}(V)(R) & \longrightarrow & \mathrm{SO}(\Lambda)(R) & \longrightarrow & R^*/(R^*)^2 \\ & & \downarrow \mathrm{SN} & \nearrow \sim & \\ & & R^*/(R^*)^2 & & \end{array}$$

The group scheme $\mathrm{SL}(V)$ is simply-connected (as its geometric fibers are semisimple algebraic group of type A_3). Thus the central isogeny $\mathrm{SL}(V)_R \rightarrow \mathrm{SO}(\Lambda)_R$ forms the universal covering of $\mathrm{SO}(\Lambda)_R$, which induces an isomorphism $\mathrm{SL}(V)_R \xrightarrow{\sim} \mathrm{Spin}(\Lambda)_R$ by using the Existence and Isomorphism Theorems over a general ring (see e.g., [16, Exercise 6.5.2]). □

Remark 4.4.2. When $R = \mathbb{Z}_\ell$, we have

$$\mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2 \cong \begin{cases} \{\pm 1\} & \text{if } \ell \neq 2, \\ \{\pm 1\} \times \{\pm 5\} & \text{if } \ell = 2. \end{cases}$$

Thus the image of $\mathrm{SL}(V)(\mathbb{Z}_\ell)$ is a finite index subgroup in $\mathrm{SO}(\Lambda)(\mathbb{Z}_\ell)$.

Remark 4.4.3. When $R = W(k)$, we have

$$W(k)^*/(W(k)^*)^2 \cong \begin{cases} \{\pm 1\} & \text{if } k = \mathbb{F}_{p^s} \text{ for } p > 2, s \geq 1 \\ \{\pm 1\} \times \{\pm 5\} & \text{if } k = \mathbb{F}_{p^s} \text{ for } p = 2, s \geq 1, \\ \{1\} & \text{if } k = \bar{k} \text{ or } k^s \subset k, \text{char}(k) > 2. \end{cases}$$

as $W(k)$ is Henselian. Thus the wedge map $\text{SL}(V)(W) \rightarrow \text{SO}(\Lambda)(W)$ is surjective when $k = \bar{k}$.

Let $V_R = \text{H}^1(X)_R$. We can see the set

$$\text{Iso}^{\text{ad},(d)}(\text{H}^1(X)_R, \text{H}^1(Y)_R)$$

is a naturally (right) $\text{SL}(V_R)$ -torsor if it is non-empty. The wedge product provides a natural map

$$\wedge^2: \text{Iso}^{\text{ad},(d)}(\text{H}^1(X)_R, \text{H}^1(Y)_R) \rightarrow \text{Iso}^{\text{ad},(d)}(\text{H}^2(X)_R, \text{H}^2(Y)_R).$$

Let $\{v_i\}$ be an admissible basis of $\text{H}^1(X)_R$ and let $\{v'_i\}$ be a d -admissible basis of $\text{H}^1(Y)_R$ respectively. There is an d -admissible isomorphism $\psi_0 \in \text{Iso}^{\text{ad},(d)}(\text{H}^1(X)_R, \text{H}^1(Y)_R)$ such that $\psi_0(v_i) = v'_i$. For a d -admissible isometry $\varphi: \text{H}^2(X, R) \rightarrow \text{H}^2(Y, R)$, we can see

$$\varphi = \wedge^2(\psi_0^{-1}) \circ g, \text{ for some } g \in \text{SO}(\Lambda_R).$$

In this way, any d -admissible isomorphism φ can be identified with (unique) element $g \in \text{SO}(\Lambda)(R)$ when the admissible bases are fixed. This allows us to deal with d -admissible isomorphisms group-theoretically. In particular, we have the following notion of spinor norm.

Definition 4.4.4. The *spinor norm* of the d -admissible isomorphism φ is defined to the image of g under $\text{SN}: \text{SO}(\Lambda)(R) \rightarrow R^*/(R^*)^2$, denoted by $\text{SN}(\varphi)$.

Lemma 4.4.5. *The spinor norm $\text{SN}(\varphi)$ is independent of the choice of admissible bases.*

Proof. For different choice of admissible bases, we can see the resulted $\tilde{g} = KgK^{-1}$ for some $K \in \text{SO}(\Lambda_R)$. Therefore $\text{SN}(\tilde{g}) = \text{SN}(g)$. \square

Remark 4.4.6. When R is a field, the spinor norm can be computed by the Cartan-Dieudonné decomposition. That means, we can write any $g \in \text{SO}(\Lambda)(R)$ as a the composition of reflections:

$$\varphi_{b_n} \circ \varphi_{b_{n-1}} \circ \cdots \circ \varphi_{b_1}$$

for some non-isotropic vectors $b_1, \dots, b_n \in \Lambda_R$, and $\text{SN}(g) = [(b_1)^2 \cdots (b_{n-1})^2 (b_n)^2]$.

Lemma 4.4.7. *The d -admissible isomorphism φ is a wedge of some d -admissible isomorphism $\psi: \text{H}^1(X, R) \rightarrow \text{H}^1(Y, R)$ if and only if $\text{SN}(\varphi) = 1$.*

Proof. The exact sequence (4.4.1) shows that if $\text{SN}(\varphi) = \text{SN}(g) = 1$, then there is some $h \in \text{SL}(V_R)$ such that $\wedge^2(h) = g$. Thus we can take $\psi = \psi_0 \circ h$ when $\text{SN}(\varphi) = 1$, and see that

$$\wedge^2(\psi) = \wedge^2(\psi_0) \circ \wedge^2(h) = \varphi.$$

The converse is clear. \square

4.5. Shioda's trick for Hodge isogenies. When $k = \mathbb{C}$ and d is an integer, we say an isometry

$$\varphi: \text{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \text{H}^2(Y, \mathbb{Q})(d)$$

a *Hodge isogeny of degree d* if it is also a morphism of Hodge structures. In particular, if $d = 1$, then it is the classical Hodge isometry we usually talk about. Clearly, a d -admissible rational Hodge isomorphism is a Hodge isogeny of degree d . In terms of spinor norms, we can generalize Shioda's theorem 4.1.1 to admissible rational Hodge isogenies.

Proposition 4.5.1 (Shioda's trick on admissible Hodge isogenies).

(1) *A d -admissible Hodge isogeny of degree d*

$$\varphi: \text{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \text{H}^2(Y, \mathbb{Q})(d)$$

is a wedge of some rational Hodge isomorphism $\psi: \text{H}^1(X, \mathbb{Q}) \xrightarrow{\sim} \text{H}^1(Y, \mathbb{Q})$, if its spinor norm is a square in \mathbb{Q}^ . In this case, the Hodge isogeny is induced by a quasi-isogeny of degree d^2 .*

(2) When $d = 1$, any admissible Hodge isometry $\varphi: \mathbb{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{H}^2(Y, \mathbb{Q})$ is induced by an isogeny $f: Y \rightarrow X$ of degree n^2 for some integer n such that $\varphi = \frac{f^*}{n}$.

Proof. Under the assumption of (1), we can find a d -admissible isomorphism ψ by applying the Lemma 4.4.7. It remains to prove that ψ preserves the Hodge structure, which is essentially the same as in [58, Theorem 1].

For (2), we shall suppose the spinor norm $\text{SN}(\varphi) = n\mathbb{Q}^{*2} \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. Let $E = \mathbb{Q}(\sqrt{n})$. We can see the base-change $\mathbb{H}^2(X, E) \xrightarrow{\sim} \mathbb{H}^2(Y, E)$ is a Hodge isometry with coefficients in E such that $\text{SN}(\varphi) = 1 \in E^*/(E^*)^2$. Then by applying Lemma 4.4.7, we will obtain an admissible (fixing admissible bases for $\mathbb{H}^1(X, \mathbb{Q})$ and $\mathbb{H}^1(Y, \mathbb{Q})$) Hodge isomorphism $\psi: \mathbb{H}^1(X, E) \xrightarrow{\sim} \mathbb{H}^1(Y, E)$. Let

$$\sigma: a + b\sqrt{n} \rightsquigarrow a - b\sqrt{n}$$

be the generator of $\text{Gal}(E/\mathbb{Q})$. As we have fixed the \mathbb{Q} -linear admissible bases, the wedge map

$$\text{SL}_4(E) \xrightarrow{\wedge^2} \text{SO}(\Lambda)(E)$$

is defined over \mathbb{Q} , and so is σ -equivariant. Let g be the element in $\text{SL}_4(E)$ corresponding to ψ . As $\wedge^2(g) \in \text{SO}(\Lambda) \subset \text{SO}(\Lambda_E)$, we can see

$$(\wedge^2(\sigma(g)) = \sigma(\wedge^2(g)) = \wedge^2(g).$$

which implies that $\sigma(g)g^{-1} = \pm \text{id}_4$ since $\ker(\wedge^2) = \{\pm \text{id}_4\}$. If $\sigma(g) = g$, then $g \in \text{SL}_4(\mathbb{Q})$ and the statement trivially holds. If $\sigma(g) = -g$, then $g_0 = \sqrt{n}g$ is lying in $\text{GL}_4(\mathbb{Q})$. Let

$$\psi_0: \mathbb{H}^1(X, \mathbb{Q}) \rightarrow \mathbb{H}^1(Y, \mathbb{Q})$$

be the corresponding element of g_0 in $\text{Iso}^{\text{ad}, (n^2)}(\mathbb{H}^1(X, \mathbb{Q}), \mathbb{H}^1(Y, \mathbb{Q}))$. As $\wedge^2\psi_0 = n\varphi$ is a Hodge isogeny, the part (1) then implies that ψ_0 is a Hodge isomorphism as well. Thus ψ_0 lifts to a quasi-isogeny $f_0: Y \rightarrow X$ and we have

$$\varphi = \wedge^2(\psi) = \frac{f_0^*}{n}: \mathbb{H}^2(X, \mathbb{Q}) \rightarrow \mathbb{H}^2(Y, \mathbb{Q}).$$

□

Remark 4.5.2. If a Hodge isometry $\psi: \mathbb{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{H}^2(Y, \mathbb{Q})$ is not admissible, i.e., its determinant is -1 with respect to some admissible bases, then we can take its composition with the isometry $\psi_{\mathcal{P}}$ induced by the Poincaré bundle as in Example 4.2.3. After that, we can see $\psi_{\mathcal{P}} \circ \psi$ is admissible and is induced by an isogeny $f: \widehat{Y} \rightarrow X$.

4.6. ℓ -adic and p -adic Shioda's trick. For the integral ℓ -adic étale cohomology, we have the following statement similar to Shioda's trick for integral Betti cohomology.

Proposition 4.6.1 (ℓ -adic Shioda's trick). *Suppose $\ell \neq 2$. For any d -admissible*

$$\varphi_{\ell}: \mathbb{H}_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_{\ell}) \xrightarrow{\sim} \mathbb{H}_{\text{ét}}^2(X_{\bar{k}}, \mathbb{Z}_{\ell}),$$

we can find a d -admissible \mathbb{Z}_{ℓ} -isomorphism ψ_{ℓ} such that $\wedge^2(\psi_{\ell}) = \varphi_{\ell}$ or $-\varphi_{\ell}$. Moreover, if φ_{ℓ} is G_k -equivariant, then ψ_{ℓ} is also G_k -equivariant after replacing k by some finite extension.

Proof. As $\mathbb{Z}_{\ell}^*/(\mathbb{Z}_{\ell}^*)^2 = \{\pm 1\}$ for any $\ell \neq 2$, the spinor/norm of φ_{ℓ} is equal to ± 1 . Thus φ_{ℓ} or $-\varphi_{\ell}$ is of spinor norm one. Then the first statement follows from Lemma 4.4.7.

Suppose φ_{ℓ} is G_k -equivariant. We may assume $\wedge^2(\psi_{\ell}) = \varphi_{\ell}$ for simplicity. For any $g \in G_k$, we have

$$\wedge^2(g^{-1}\psi_{\ell}g) = g^{-1}\wedge^2(\psi_{\ell})g = \varphi_{\ell} = \wedge^2(\psi_{\ell}).$$

Therefore, $g^{-1}\psi_{\ell}g = \pm\psi_{\ell}$. By passing to a finite extension k'/k , we always have $g^{-1}\psi_{\ell}g = \psi_{\ell}$ for all $g \in G_{k'}$ which proves the assertion. □

For F -crystals attached to abelian surfaces, we can also play Shioda's trick.

Proposition 4.6.2 (*p*-adic Shioda's trick). *Suppose k is a finite field \mathbb{F}_{p^s} with odd prime p . For any d -admissible W -linear isomorphism*

$$\varphi_W : H_{\text{crys}}^2(Y/W) \xrightarrow{\sim} H_{\text{crys}}^2(X/W),$$

we can find a d -admissible W -linear isomorphism $\rho : H_{\text{crys}}^1(Y/W) \xrightarrow{\sim} H_{\text{crys}}^1(X/W)$ such that $\wedge^2(\rho) = \varphi_W$ or $-\varphi_W$. Moreover, if φ_W is a morphism of F -crystals, then ρ is an isomorphism of 2nd-iterate of F -crystals.

Proof. The first statement follows from a similar reason as in Proposition 4.6.1 as $W^*/(W^*)^2 = \{\pm 1\}$ (see Remark 4.4.3).

For the second statement, we assume $\wedge^2(\rho) = \varphi_W$. If φ_W commutes with the Frobenius action, then we have

$$\wedge^2(C_X^{-1} \cdot \rho^{(1)} \cdot C_Y) = \varphi_W.$$

as in §4.3. Thus $C_X^{-1} \cdot \rho^{(1)} \cdot C_Y = \pm \rho$, which implies

$$\rho \circ F_X = \pm F_Y \circ \rho$$

by Lemma 4.3.1. Therefore, ρ commutes with the 2nd-iterate Frobenius F_X^2 and F_Y^2 . \square

Remark 4.6.3. If k is an algebraically closed field or the separable closure in an algebraic closure such that $\text{char}(k) > 2$, then Proposition 4.6.2 also holds. In addition, the first statement can be enforced to $\wedge^2(\rho) = \varphi_W$; see Remark 4.4.3.

Combined with Tate's isogeny theorem, we have the following direct consequences of Propositions 4.6.1 and 4.6.2. It includes a special case of Tate's conjecture.

Corollary 4.6.4. *Suppose k is a finitely generated field over \mathbb{F}_p with p an odd prime. Let $\ell \neq 2$ be a prime not equal to p .*

(1) *For any admissible isometry of $\text{Gal}(k^s/k)$ -modules*

$$\varphi_\ell : H_{\text{ét}}^2(Y_{k^s}, \mathbb{Z}_\ell) \xrightarrow{\sim} H_{\text{ét}}^2(X_{k^s}, \mathbb{Z}_\ell),$$

we can find a \mathbb{Z}_ℓ -quasi-isogeny $f_\ell \in \text{Hom}_{k'}(X_{k'}, Y_{k'}) \otimes \mathbb{Z}_\ell$ for some finite extension k'/k , which induces φ_ℓ or $-\varphi_\ell$. In particular, φ_ℓ is algebraic.

(2) *For any admissible isometry of F -crystals over the Cohen ring W*

$$\varphi_W : H_{\text{crys}}^2(Y/W) \xrightarrow{\sim} H_{\text{crys}}^2(X/W),$$

we can find a \mathbb{Z}_{p^2} -quasi-isogeny $f_p \in \text{Hom}_{k'}(X_{k'}, Y_{k'}) \otimes \mathbb{Z}_{p^2}$ which induces φ_W or $-\varphi_W$ for some finite extension k'/k , where $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$. In particular, φ_W is algebraic.

Proof. For (1), Proposition 4.6.1 implies there is an $\text{Gal}(k^s/k)$ -equivariant isomorphism

$$\psi_\ell : H_{\text{ét}}^1(Y_{k^s}, \mathbb{Z}_\ell) \xrightarrow{\sim} H_{\text{ét}}^1(X_{k^s}, \mathbb{Z}_\ell),$$

inducing φ_ℓ or $-\varphi_\ell$ after a finite extension of k . Then f_ℓ exists by the canonical bijection (cf. [20, VI, §3 Theorem 1])

$$\text{Hom}_k(X, Y) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \text{Hom}_{\text{Gal}(k^s/k)}(H_{\text{ét}}^1(Y_{k^s}, \mathbb{Z}_\ell), H_{\text{ét}}^1(X_{k^s}, \mathbb{Z}_\ell)).$$

For (2), let \bar{k} be an algebraic closure of k , then Proposition 4.6.2 and Remark 4.6.3 imply that there is an isomorphism

$$\rho : H_{\text{crys}}^1(Y_{\bar{k}}/W(\bar{k})) \xrightarrow{\sim} H_{\text{crys}}^1(X_{\bar{k}}/W(\bar{k}))$$

such that $F_{X_{\bar{k}}} \circ \rho = \pm \rho \circ F_{Y_{\bar{k}}}$. In fact, the \bar{k} in this formula can be replaced a finite extension k' of k by a similar argument as the proof of (2) of Proposition 4.5.1.

Replace k by k' . If $F_X \circ \rho = \rho \circ F_Y$ then one can conclude by the canonical isomorphisms

$$\text{Hom}_k(X, Y) \otimes \mathbb{Z}_p \xrightarrow{\sim} \text{Hom}_k(X[p^\infty], Y[p^\infty]) \xrightarrow{\sim} \text{Hom}_F(H^1(Y/W), H^1(X/W)), \quad (4.6.1)$$

where the bijectivity of the first arrow is given by p -adic Tate's isogeny theorem (cf. [18, Theorem 2.6]) and the second one is the faithfulness of taking Dieudonné module over W (cf. [17, Theorem]).

It remains to consider the case $F_X \circ \rho = -\rho \circ F_Y$. After taking a finite extension of k , we may assume that $\mathbb{Z}_{p^2} \subset W(k)$. Now there is $\xi \in W(k)$ such that $\xi^{p-1} + 1 = 0$. We can see that

$$F_X \circ (\xi\rho) = \xi^p F_X \circ \rho = (\xi\rho) \circ F_Y.$$

Again, the bijection (4.6.1) implies that $\xi\rho$ is induced by a \mathbb{Z}_p -quasi-isogeny $f_0 \in \text{Hom}_k(X, Y) \otimes \mathbb{Z}_p$. Note that $\xi \in \mathbb{Z}_{p^2}^*$. We can take

$$f_p = \frac{f_0}{\xi} \in \text{Hom}_k(X, Y) \otimes \mathbb{Z}_{p^2}.$$

□

Remark 4.6.5. In [67], Zarhin introduces the notion of *almost isomorphism*. Two abelian varieties over k are called almost isomorphic if their Tate modules T_ℓ are isomorphic as Galois modules (replaced by p -divisible groups when $\ell = p$). Proposition 4.6.1 and 4.6.2 imply that it is possible to characterize almost isomorphic abelian surfaces by their 2nd-cohomology groups.

5. DERIVED ISOGENY IN CHARACTERISTIC ZERO

In this section, we follow [23] and [31] to prove the twisted Torelli theorem for abelian surfaces over algebraically closed fields of characteristic zero.

5.1. Over \mathbb{C} : Hodge isogeny versus derived isogeny. Let X and Y be complex abelian surfaces.

Definition 5.1.1. A rational Hodge isometry $\psi_b: \mathbb{H}^2(X, \mathbb{Q}) \rightarrow \mathbb{H}^2(Y, \mathbb{Q})$ is called *reflexive* if it is induced by a reflection on Λ along a vector $b \in \Lambda$:

$$\varphi_b: \Lambda_{\mathbb{Q}} \xrightarrow{\sim} \Lambda_{\mathbb{Q}} \quad x \mapsto x - \frac{2(x, b)}{(b, b)}b.$$

Lemma 5.1.2. Any reflexive Hodge isometry ψ_b induces a Hodge isometry on twisted Mukai lattices

$$\tilde{\psi}_b: \tilde{\mathbb{H}}(X, \mathbb{Z}; B) \rightarrow \tilde{\mathbb{H}}(Y, \mathbb{Z}; B'),$$

where $B = \frac{2}{(b, b)}b \in \mathbb{H}^2(X, \mathbb{Q})$ (via some marking $\Lambda \cong \mathbb{H}^2(X, \mathbb{Z})$) and $B' = -\psi_b(B)$.

Proof. The proof can be found in [31, §1.2]. □

In analogy to [31, Theorem 1.1], the following result characterizes the reflexive Hodge isometries between abelian surfaces.

Theorem 5.1.3. Let X and Y be two complex abelian surfaces. If there is a reflexive Hodge isometry

$$\psi_b: \mathbb{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{H}^2(Y, \mathbb{Q}),$$

for some $b \in \Lambda$, then there exist $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$ such that ψ_b is induced by a derived equivalence

$$\mathbb{D}^b(X, \alpha) \simeq \mathbb{D}^b(Y, \beta).$$

Equivalently, X or \hat{X} is isomorphic to the coarse moduli space of twisted coherent sheaves on Y , and ψ_b is induced by the twisted Fourier-Mukai transform associated to the universal twisted sheaf.

Proof. According to Lemma 5.1.2, there is a Hodge isometry

$$\tilde{\psi}_b: \tilde{\mathbb{H}}(X, \mathbb{Z}; B) \xrightarrow{\sim} \tilde{\mathbb{H}}(Y, \mathbb{Z}; B').$$

Let $v_{B'}$ be the image of Mukai vector $(0, 0, 1)$ under $\tilde{\psi}_b$. From our construction, there is a Mukai vector

$$v = \exp(-B') \cdot v_{B'} \in \tilde{\mathbb{H}}(Y, \mathbb{Z})$$

satisfying $v_{B'} = \exp(B') \cdot v$. We can assume that v is positive (see Proposition 3.4.1) by some suitable autoequivalences of $\mathbb{D}^b(Y)$ as in [34, §2]. Let β be the Brauer class on Y with respect to B' and $\mathcal{Y} \rightarrow Y$ be the corresponding \mathbb{G}_m -gerbe. For some $v_{B'}$ -generic polarization H , the

moduli stack $\mathcal{M}_H(\mathcal{Y}, v_{B'})$ of β -twisted sheaves on Y with Mukai vector $v_{B'}$ forms a \mathbb{G}_m -gerbe on its coarse moduli space $M_H(\mathcal{Y}, v_{B'})$ such that

$$[\mathcal{M}_H(\mathcal{Y}, v_{B'})] \in \mathrm{Br}(M_H(\mathcal{Y}, v_{B'})) [r]$$

(cf. [39, Proposition 2.3.3.4, Corollary 2.3.3.7]).

The kernel \mathcal{P} is the tautological twisted sheaf on $\mathcal{Y} \times \mathcal{M}_H(\mathcal{Y}, v_{B'})$ which induces a twisted Fourier-Mukai transform

$$\Phi_{\mathcal{P}}: D^b(Y, \beta) \rightarrow D^b(\mathcal{M}_H(\mathcal{Y}, v_{B'})) \simeq D^b(M_{H'}(\mathcal{Y}, v_{B'}), \alpha),$$

where $\alpha = [\mathcal{M}_H(\mathcal{Y}, v_{B'})] \in \mathrm{Br}(M_{H'}(\mathcal{Y}, v_{B'}))$ (cf. [66, Theorem 4.3]). It induces a Hodge isometry

$$\tilde{H}(Y, \mathbb{Z}; B') \xrightarrow{\sim} \tilde{H}(M_H(\mathcal{Y}, v_{B'}), \mathbb{Z}; B''),$$

where B'' is a \mathbf{B} -field lift of α . Its composition with $\tilde{\psi}_b$ is a Hodge isometry

$$\tilde{H}(X, \mathbb{Z}; B) \xrightarrow{\sim} \tilde{H}(M_H(\mathcal{Y}, v_{B'}), \mathbb{Z}; B''), \quad (5.1.1)$$

sending the Mukai vector $(0, 0, 1)$ to $(0, 0, 1)$ and preserving the Mukai pairing. We can see $(1, 0, 0)$ is mapping to $(1, b, \frac{b^2}{2})$ for some $b \in H^2(Y, \mathbb{Z})$ via (5.1.1). Thus we can replace B'' by $B'' + b$, which will not change the corresponding Brauer class, to obtain a Hodge isometry which takes $(1, 0, 0)$ to $(1, 0, 0)$ and $(0, 0, 1)$ to $(0, 0, 1)$ at the same time. This yields a Hodge isometry

$$H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(M_{H'}(\mathcal{Y}, v_{B'}), \mathbb{Z}).$$

Then we can apply Shioda's Torelli Theorem of abelian surfaces [58] to conclude that

$$M_{H'}(\mathcal{Y}, v_{B'}) \cong X \text{ or } \hat{X}.$$

When $X \cong M_{H'}(\mathcal{Y}, v_{B'})$, $\Phi_{\mathcal{P}}$ gives the derived equivalence as desired. When $\hat{X} \cong M_{H'}(\mathcal{Y}, v_{B'})$, we can prove the assertion by using the fact X and \hat{X} are derived equivalent. \square

Next, we are going to show that any rational Hodge isometry can be decomposed into a chain of reflexive Hodge isometries. This is a special case of Cartan-Dieudonné theorem which says that any element $\varphi \in \mathrm{SO}(\Lambda_{\mathbb{Q}})$ can be decomposed as products of reflections:

$$\varphi = \varphi_{b_1} \circ \varphi_{b_2} \circ \cdots \circ \varphi_{b_n}, \quad (5.1.2)$$

such that $b_i \in \Lambda$, and $(b_i)^2 \neq 0$. Then from the surjectivity of period map [58, Theorem II], for any rational Hodge isometry

$$H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q}),$$

we can find a sequence of abelian surfaces $\{X_i\}$ with Λ -markings and Hodge isometries $\psi_{b_i}: H^2(X_{i-1}, \mathbb{Q}) \xrightarrow{\sim} H^2(X_i, \mathbb{Q})$, where $X_0 = X$ and $X_n = Y$, such that ψ_{b_i} induces φ_{b_i} on $\Lambda_{\mathbb{Q}}$. We can arrange them as (1.1.1):

$$\begin{aligned} H^2(X, \mathbb{Q}) &\xrightarrow{\psi_{b_1}} H^2(X_1, \mathbb{Q}) \\ &\xrightarrow{\psi_{b_2}} H^2(X_2, \mathbb{Q}) \\ &\vdots \\ &\xrightarrow{\psi_{b_n}} H^2(Y, \mathbb{Q}). \end{aligned} \quad (5.1.3)$$

Finally, this yields

Corollary 5.1.4. *If there is a rational Hodge isometry $\varphi: H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q})$, then there is a derived isogeny from X to Y , whose Hodge realization is φ .*

As a consequence, we get

Corollary 5.1.5. *There is a rational Hodge isometry $H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q})$ if and only if there is a derived isogeny from $\mathrm{Km}(X)$ to $\mathrm{Km}(Y)$.*

Proof. Witt's cancellation theorem implies that

$$H^2(X, \mathbb{Q}) \simeq H^2(Y, \mathbb{Q}) \Leftrightarrow T(X)_{\mathbb{Q}} \simeq T(Y)_{\mathbb{Q}},$$

as Hodge isometries, where $T(-)$ denotes the transcendental part of $H^2(-)$. According to [31, Theorem 0.1], $\text{Km}(X)$ is derived isogenous to $\text{Km}(Y)$ if and only if there is a Hodge isometry $T(\text{Km}(X))_{\mathbb{Q}} \simeq T(\text{Km}(Y))_{\mathbb{Q}}$. Then the statement is clear from the fact there is a canonical integral Hodge isometry $T(X)(2) \simeq T(\text{Km}(X))$ (cf. [47, Proposition 4.3(i)]). \square

Remark 5.1.6. A consequence of Corollary 5.1.4 is that any rational Hodge isometry between abelian surfaces is algebraic, which is a special case of Hodge conjecture on product of two abelian surface. Unlike the case of K3 surfaces, the Hodge conjecture for product of abelian surfaces were known for a long time. See [56, Theorem 3.15] for example.

Moreover, we may call a reflexive Hodge isometry

$$\psi_b: H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q})$$

induced by a primitive vector $b \in \Lambda$ *prime-to- ℓ* if $\ell \nmid n = \frac{(b)^2}{2}$. The following results imply that the Hodge realization of prime-to- ℓ derived isogeny is a composition of finitely many prime-to- ℓ reflexive Hodge isometries.

Lemma 5.1.7. *If the induced derived isogeny $D^b(X) \sim D^b(Y)$ in Corollary 5.1.4 is prime-to- ℓ , then each reflexive Hodge isometry ψ_b in (5.1.3) is prime-to- ℓ .*

Proof. Otherwise, we can take ℓ^k to be the ℓ -factor of n . As ψ_b restricts to an isomorphism

$$H^2(X, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} H^2(Y, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)},$$

we have $\ell^k \mid (x, b)$ for any $x \in \Lambda$. This means ℓ^k divides the divisibility of b , which is impossible as Λ is unimodular. \square

Remark 5.1.8. With notations in Theorem 5.1.3, if ψ_b is prime-to- ℓ with $n = \frac{(b)^2}{2}$, then the Fourier-Mukai equivalence $D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$ in Theorem 5.1.3 satisfies

$$\alpha^n = \exp(nB) = 1 \in \text{Br}(X),$$

which implies $\alpha \in \text{Br}(X)[n]$. Similarly, n divides the order of $\beta = \exp(B') \in \text{Br}(Y)$.

5.2. Isogeny versus derived isogeny. Let us now describe derived isogenies via suitable isogenies.

Recall that the isogeny category of abelian varieties $\text{AV}_{\mathbb{Q}, k}$ consists of all abelian varieties over a field k as objects, and the Hom-sets are

$$\text{Hom}_{\text{AV}_{\mathbb{Q}, k}}(X, Y) := \text{Hom}_{\text{AV}_k}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\text{Hom}_{\text{AV}_k}(X, Y)$ is the abelian group of homomorphisms from X to Y with the natural addition. We may also write $\text{Hom}^0(X, Y)$ for $\text{Hom}_{\text{AV}_{\mathbb{Q}, k}}(X, Y)$ if there are no confusion on the field of definition k . An isomorphism f from X to Y in the isogeny category $\text{AV}_{\mathbb{Q}, k}$ is called a quasi-isogeny from X to Y . For any quasi-isogeny f , we can find a minimal integer n such that

$$nf: X \rightarrow Y$$

is an isogeny, i.e., a finite surjective morphism of abelian varieties. When $k = \mathbb{C}$, with the uniformization of complex abelian varieties, we have a canonical bijection

$$\text{Hom}_{\text{AV}_{\mathbb{Q}, \mathbb{C}}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\text{Hdg}}(H^1(Y, \mathbb{Q}), H^1(X, \mathbb{Q})),$$

where the right-hand side is the set of \mathbb{Q} -linear morphisms preserving Hodge structures. Then the integer n for f is also the minimal integer such that $(nf)^*(H^1(Y, \mathbb{Z})) \subseteq H^1(X, \mathbb{Z})$.

It is well-known that the functor $\underline{\text{Hom}}(X, Y)$ of homomorphisms from X to Y (not just as scheme morphisms) is representable by an étale group scheme over k (see [62, (7.14)] for example). Therefore, via Galois descent, we have

$$\text{Hom}_{\text{AV}_{\bar{k}}}(X_{\bar{k}}, Y_{\bar{k}}) \xrightarrow{\sim} \text{Hom}_{\text{AV}_{\bar{K}}}(X_{\bar{K}}, Y_{\bar{K}}), \quad (5.2.1)$$

for any algebraically closed field $\bar{K} \supset k$. A similar statement holds for derived isogenies.

Lemma 5.2.1. *Let X and Y be abelian surfaces defined over k with $\text{char}(k) = 0$. Let $\bar{K} \supseteq k$ be an algebraically closed field containing k . Let \bar{k} be the algebraic closure of k in \bar{K} . Then if $X_{\bar{K}}$ and $Y_{\bar{K}}$ are twisted derived equivalent, so is $X_{\bar{k}}$ and $Y_{\bar{k}}$.*

Proof. As $X_{\bar{K}}$ is twisted derived equivalent to $Y_{\bar{K}}$, by Theorem 3.4.5, there exist finitely many abelian surfaces X_0, X_1, \dots, X_n defined over \bar{K} with $X_0 = X_{\bar{K}}$ and

$$X_i \text{ or } \widehat{X}_i = M_{H_i}(\mathcal{X}_{i-1}, v_i) \quad Y_{\bar{K}} \text{ or } \widehat{Y}_{\bar{K}} \cong M_{H_n}(\mathcal{X}_n, v_n)$$

for some $[\mathcal{X}_{i-1}] \in \text{Br}(X_{i-1})$. Let us construct abelian surfaces over \bar{k} to connect $X_{\bar{k}}$ and $Y_{\bar{k}}$ as follows:

Set $X'_0 = X_{\bar{k}}$, then we take $X'_1 = M_{H'_1}(\mathcal{X}'_0, v'_1)$ where \mathcal{X}'_0, H'_1 and v'_1 are the descent of \mathcal{X}_0, H_1 and v via the isomorphisms $\text{Br}(X_{\bar{K}}) \cong \text{Br}(X_{\bar{k}})$, $\text{Pic}(X_{\bar{K}}) \cong \text{Pic}(X_{\bar{k}})$ and $\widetilde{H}(X_{\bar{K}}) \cong \widetilde{H}(X_{\bar{k}})$. Then inductively, we can define X'_i as the moduli space of twisted sheaves $M_{H'_i}(\mathcal{X}'_{i-1}, v'_i)$ (or its dual respectively) over \bar{k} . Note that we have natural isomorphisms

$$(M_{H'_i}(\mathcal{X}'_{i-1}, v'_i))_{\bar{K}} \cong M_{H_i}(\mathcal{X}_{i-1}, v_i)$$

over \bar{K} . In particular, $(M_{H'_i}(\mathcal{X}'_n, v'_i))_{\bar{K}} \cong Y_{\bar{K}}$. It follows that $M_{H'_i}(\mathcal{X}'_n, v'_i) \cong Y_{\bar{k}}$. \square

More generally, we can replace \mathbb{Q} in $\text{AV}_{\mathbb{Q},k}$ by any ring R . Then any isomorphism from X to Y in $\text{AV}_{R,k}$ will be called a R -(quasi)-isogeny. In particular,

Definition 5.2.2. An element $f \in \text{Hom}_k(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$ which has an inverse in $\text{Hom}_k(Y, X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$ is called a prime-to- ℓ quasi-isogeny, where $\mathbb{Z}_{(\ell)}$ is the localization of \mathbb{Z} at (ℓ) .

For any abelian surface $X_{\mathbb{C}}$ over \mathbb{C} , the spreading out argument shows that there is a finitely generated field $k \subset \mathbb{C}$ and an abelian surface X over k such that $X \times_k \mathbb{C} \cong X_{\mathbb{C}}$. We have the following Artin comparison

$$H_{\text{ét}}^i(X_{\bar{k}}, \mathbb{Z}_{(\ell)}) \cong H^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}, \quad (5.2.2)$$

for any $i \in \mathbb{Z}$ and ℓ a prime. Suppose Y is another abelian surface defined over k . Suppose $f: Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is a prime-to- ℓ quasi-isogeny. By definition, it induces an isomorphism of $\mathbb{Z}_{(\ell)}$ -modules

$$f^*: H^1(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} \xrightarrow{\sim} H^1(Y_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)},$$

such that there is a commutative diagram

$$\begin{array}{ccc} H^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} & \xrightarrow{\sim} & H^i(Y_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} \\ \downarrow & & \downarrow \\ H_{\text{ét}}^i(X_{\bar{k}}, \mathbb{Z}_{(\ell)}) & \xrightarrow{\sim} & H_{\text{ét}}^i(Y_{\bar{k}}, \mathbb{Z}_{(\ell)}) \end{array}$$

for any i , under the comparison (5.2.2). For the converse, we have the following simple fact given by a faithfully flat descent of modules along $\mathbb{Z}_{(\ell)} \hookrightarrow \mathbb{Z}_{\ell}$ and the ℓ -adic Shioda thick.

Lemma 5.2.3. *A (quasi)-isogeny $f: Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is prime-to- ℓ if and only if it induces an isomorphism of integral ℓ -adic realizations*

$$f^*: H_{\text{ét}}^2(X_{\bar{k}}, \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_{(\ell)}).$$

Inspired by Shioda's trick for Hodge isogenies 4.5.1, we introduce the following notions.

Definition 5.2.4. Let X and Y be g -dimensional abelian varieties over k . We say X and Y are (prime-to- ℓ) *principally isogenous* if there is a (prime-to- ℓ) isogeny f from X or \widehat{X} to Y of square degree, i.e., $\deg(f) = d^2$ for some $d \in \mathbb{Z}$. In this case, we may call f a *principal isogeny*.

Furthermore, we say f is *quasi-liftable* if f can be written as the composition of finitely many isogenies that are liftable to characteristic zero.

Now, we can state the main result in this section.

Theorem 5.2.5. *Suppose $\text{char}(k) = 0$. The following statements are equivalent:*

- (1) X is (prime-to- ℓ) principally isogenous to Y over \bar{k} .

(2) X and Y are (prime-to- ℓ) derived isogenous over \bar{k} .

Proof. (1) \Rightarrow (2): we can assume that $f : X \rightarrow Y$ is a principal isogeny defined over a finitely generated field k' . By embedding k' into \mathbb{C} , two complex abelian surfaces $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are derived isogenous since there is a rational Hodge isometry

$$\frac{1}{n}f^* \otimes \mathbb{Q} : H^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q} \cong H^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q}$$

where $\deg(f) = n^2$. By Lemma 5.2.1, one can conclude $X_{\bar{k}}$ and $Y_{\bar{k}}$ are derived isogenous, with the rational Hodge realization $\frac{1}{n}f^* \otimes \mathbb{Q}$.

If f is a prime-to- ℓ isogeny, the map $\frac{1}{n}f^*$ restricts to an isomorphism

$$H^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} H^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)}.$$

Then one can take a prime-to- ℓ Cartan-Dieudonné decomposition (see Lemma 5.2.6 below), which decomposes $\frac{1}{n}f^* \otimes \mathbb{Q}$ into a sequence of prime-to- ℓ reflexive Hodge isometries. The assertion follows immediately.

To deduce (2) \Rightarrow (1), we may assume X and Y are derived isogenous over a finitely generated field k' . Embed k' into \mathbb{C} , $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are derived isogenous as well. According to Proposition 4.5.1, they are principally isogenous over \mathbb{C} . It follows that X and Y are principally isogenous over \bar{k} by Lemma (5.2.1).

If $D^b(X) \sim D^b(Y)$ is prime-to- ℓ , then each reflexive Hodge isometry ψ_b in (5.1.3) is prime-to- ℓ by Lemma 5.1.7. The principal isogeny which induces ψ_b is prime-to- ℓ by Lemma 5.2.3. This proves the assertion. \square

Lemma 5.2.6 (prime-to- ℓ Cartan-Dieudonné decomposition). *Let Λ be an integral lattice over \mathbb{Z} whose reduction mod ℓ is still non-degenerated. Any orthogonal matrix $A \in O(\Lambda)(\mathbb{Z}_{(\ell)}) \subset O(\Lambda)(\mathbb{Q})$, with $(\ell > 2)$, can be decomposed into a sequence of prime-to- ℓ reflections.*

Proof. To prove the assertion, we will follow the proof of [57] to refine Cartan-Dieudonné decomposition for any field. In general, if Λ_k is quadratic space over a field k with the Gram matrix G . Let I be the identity matrix and let R_b be the reflection with respect to $b \in \Lambda_k$. The proof of Cartan-Dieudonné decomposition in [57] relies on the following facts: for any element $A \in O(\Lambda_k)$, we have

- i) A is a reflection if $\text{rank}(A - I) = 1$ (cf. [57, Lemma 2])
- ii) if $S = G(A - I)$ is not skew symmetric, there exists $a \in \Lambda$ satisfying $a^t S a \neq 0$ and

$$S + S^t \neq \frac{1}{a^t S a} (S b \cdot b^t S + S^t b \cdot b^t S^t),$$

then $\text{rank}(A R_b - I) = \text{rank}(A - I) - 1$ and $G(A R_b - I)$ is not skew symmetric with $b = (A - I)a$ satisfying $b^2 = -2a^t S a$. Such a always exists. (cf. [57, Lemma 4, Lemma 5]).

- iii) if $S = G(A - I)$ is skew symmetric, then there exists $b \in \Lambda$ such that $G(A R_b - I)$ is not skew symmetric (cf. [57, Theorem 2]).

Then one can decompose A as a series of reflections by repeatedly using ii). In our case, it suffices to show that if A is coprime to ℓ , i.e. nA is integral for some n coprime to ℓ , then

- i') A is a coprime to ℓ reflection if $\text{rank}(A - I) = 1$;
- ii') if $S = G(A - I)$ is not skew symmetric and there exists $a \in \Lambda$ satisfying $p \nmid a^t S a$ and

$$S + S^t \neq \frac{1}{a^t S a} (S b \cdot b^t S + S^t b \cdot b^t S^t),$$

then $A R_b$ is coprime to ℓ and $G(A R_b - I)$ is not skew symmetric with b constructed above;

- iii') if $S = G(A - I)$ is skew symmetric, then there exists $b \in \Lambda$ such that $A R_b$ is coprime to ℓ and $G(A R_b - I)$ is not skew symmetric.

This means that we only need to find some prime-to- ℓ reflections satisfying the conditions as above. By our assumption, the modulo ℓ reduction $\Lambda_{\mathbb{F}_\ell}$ of Λ remains non-degenerate. If A is coprime to ℓ , then we can consider the reduction $A \bmod \ell$ and apply i)-iii) to $A \bmod \ell \in \mathcal{O}(\Lambda_{\mathbb{F}_\ell})$ to obtain reflections over \mathbb{F}_ℓ . We can lift the reflections to $\mathcal{O}(\Lambda_{\mathbb{Q}})$, which are obviously coprime to ℓ . One can easily check such reflections satisfy ii') and iii'). \square

5.3. Proof of Theorem 1.2.1 and Corollary 1.2.2. Let us summarize all the results which conclude Theorem 1.2.1. By a similar argument in Theorem 5.2.5, we can reduce them to the case $k = \mathbb{C}$.

Proof of (i) \Leftrightarrow (ii). This is Theorem 5.2.5.

Proof of (i) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii). The equivalence (i) \Leftrightarrow (vi) is Corollary 5.1.4. The equivalence (vi) \Leftrightarrow (vii) is from Witt cancellation theorem. For (vi) \Leftrightarrow (viii), note that a rational Hodge isometry $\varphi: H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(Y, \mathbb{Q})$ induces a rational isometry $\mathrm{NS}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{NS}(Y)_{\mathbb{Q}}$. Then we have a Hodge isometry $T(X)_{\mathbb{Q}} \xrightarrow{\sim} T(Y)_{\mathbb{Q}}$ by Witt cancellation theorem. The converse is clear.

Proof of (i) \Leftrightarrow (iii). This is Corollary 5.1.5.

Proof of (ii) \Rightarrow (iv) \Rightarrow (v). This is from the computation in [23, Proposition 4.6]. Indeed, one may take the correspondence

$$\Gamma := \bigoplus_i \Gamma_{2i}: \mathfrak{h}^{\mathrm{even}}(X) \xrightarrow{\sim} \mathfrak{h}^{\mathrm{even}}(Y),$$

where

$$\Gamma_{2i} := \frac{1}{n^i} f^* \circ \pi_X^{2i}: \mathfrak{h}^{2i}(X) \rightarrow \mathfrak{h}^{2i}(Y),$$

and $f: X \rightarrow Y$ is the given principal isogeny.

Proof of (v) \Rightarrow (ii). Let $\Gamma: \mathfrak{h}^{\mathrm{even}}(X) \xrightarrow{\sim} \mathfrak{h}^{\mathrm{even}}(Y)$ be the isomorphism of Frobenius algebra objects. The Betti realization of its second component is a Hodge isometry by the Frobenius condition (cf. [23, Theorem 3.3]). Thus X and Y are derived isogenous by Corollary 5.1.4, and hence principally isogenous.

6. DERIVED ISOGENY IN POSITIVE CHARACTERISTIC

In this section, we will prove the twisted derived Torelli theorem for abelian surfaces over odd characteristic fields. The principal strategy is to lift everything to characteristic zero.

6.1. Prime-to- p derived isogeny in mixed characteristic. Let us start with an important lemma for prime-to- p derived isogenies, which is the only place we require the characteristic $p > 3$.

Lemma 6.1.1. *Let K be a complete discretely valued field in characteristic zero, whose residue field is perfect with characteristic $p > 3$. Denote by \mathcal{O}_K the ring of integers. Let $\mathfrak{X} \rightarrow \mathcal{X}$ and $\mathfrak{Y} \rightarrow \mathcal{Y}$ be twisted abelian surfaces over \mathcal{O}_K whose special fibers are $\mathcal{X}_0 \rightarrow X_0$ and $\mathcal{Y} \rightarrow Y_0$, and generic fibers are $\mathfrak{X}_K \rightarrow \mathcal{X}_K$ and $\mathfrak{Y}_K \rightarrow \mathcal{Y}_K$. Suppose $f_0: D^b(\mathcal{X}_0) \xrightarrow{\sim} D^b(\mathcal{Y}_0)$ is a prime-to- p derived equivalence and $f: D^b(\mathfrak{X}) \xrightarrow{\sim} D^b(\mathfrak{Y})$ is a lifting of f_0 , then $f_K: D^b(\mathfrak{X}_K) \xrightarrow{\sim} D^b(\mathfrak{Y}_K)$ is also prime-to- p .*

Proof. It suffices to prove that the p -adic realization of f_K is integral. This can be deduced from an argument from Cais–Liu’s crystalline cohomological description for integral p -adic Hodge theory (cf. [13]).

Let us sketch the proof. As f_0 is prime-to- p , its cohomological realization restricts to an isometry of F -crystals

$$\varphi_p: H_{\mathrm{crys}}^2(X_0/W) \simeq H_{\mathrm{crys}}^2(Y_0/W)$$

by our definition. The base-extension $\varphi_p \otimes K$ can be identified with the cohomological realization of f_K on the de Rham cohomology

$$\varphi_K: H_{\text{dR}}^2(\mathcal{X}_K/K) \simeq H_{\text{dR}}^2(\mathcal{Y}_K/K)$$

by Berthelot–Ogus comparison (*cf.* [24, Theorem B.3.1]). It also preserves the Hodge filtration. Let S be the divided power envelope of pair $(W[[u]], \ker(W[[u]] \rightarrow \mathcal{O}_K))$. Then the map

$$\varphi_p \otimes_W S: H_{\text{crys}}^2(X_0/S) \xrightarrow{\sim} H_{\text{crys}}^2(Y_0/S)$$

is an isomorphism of strongly divisible S -lattices (*cf.* [13, §4]). If $p > 3$, then according to [13, Theorem 5.4], one can apply Breuil’s functor T_{st} on it to see that φ_K restricts to an \mathbb{Z}_p -integral $\text{Gal}(\bar{K}/K)$ -equivariant isometry

$$H_{\text{ét}}^2(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p) \xrightarrow{\sim} H_{\text{ét}}^2(\mathcal{Y}_{\bar{K}}, \mathbb{Z}_p).$$

□

Remark 6.1.2. The technical requirement for $p > 3$ is needed in [13, Theorem 4.3 (3),(4)]. When $\mathcal{O}_K = W(k)$ is unramified, this condition can be released to $p > 2$ by using Fontaine’s result [21, Theorem 2 (iii)]. In general, when $p = 3$, a possible approach is to prove the Shioda’s trick as in §4 for strongly divisible S -lattices (*cf.* [10, Definition 2.1.1]), which can reduce the statement to crystalline Galois representations of Hodge–Tate weight one.

6.2. Serre–Tate theory and lifting of prime-to- p quasi-isogeny. The Serre–Tate theorem says that the deformation theory of an abelian scheme in characteristic p is equivalent to the deformation theory of its p -divisible group (*cf.* [44, Chapter V.§2, Theorem 2.3]). Let $S_0 \hookrightarrow S$ be an infinitesimal thickening of schemes such that p is locally nilpotent on S . Let $\mathcal{D}(S_0, S)$ be the category of pairs $(\mathcal{X}_0, \mathcal{G})$, where \mathcal{X}_0 is an abelian scheme over S_0 and \mathcal{G} is a lifting of p -divisible group $\mathcal{X}_0[p^\infty]$ to S . There is an equivalence of categories

$$\begin{aligned} \{\text{abelian schemes over } S\} &\xrightarrow{\sim} \mathcal{D}(S_0, S) \\ \mathcal{X} &\mapsto (\mathcal{X} \times_S S_0, \mathcal{X}[p^\infty]). \end{aligned}$$

Now we set $S_0 = \text{Spec}(k)$ and $S = \text{Spec}(V/(\pi^{n+1}))$ for a perfect field k , V is a totally ramified finite extension of $W(k)$ and an integer $n \geq 1$. Since there is an equivalence between the category of p -divisible groups over V and the category of inductive systems of p -divisible groups over $V/(\pi^n)$, we have an identification

$$\mathcal{D}(k, V) = \varprojlim_n \mathcal{D}(k, V/(\pi^n)).$$

As a consequence, we get

Lemma 6.2.1. *There is an equivalence of categories*

$$\begin{aligned} \{\text{formal abelian schemes over } V\} &\xrightarrow{\sim} \mathcal{D}(k, V) \\ A &\mapsto (A \times_V k, A[p^\infty]). \end{aligned}$$

One can lift separable isogenies between abelian surfaces, which is well-known to experts.

Proposition 6.2.2. *Suppose $p > 2$. Let $f: X \rightarrow Y$ be a separable isogeny. There are liftings $\mathcal{X} \rightarrow \text{Spec}(V)$ and $\mathcal{Y} \rightarrow \text{Spec}(V)$ of X and Y respectively, such that the isogeny f can be lifted to an isogeny $f_V: \mathcal{X} \rightarrow \mathcal{Y}$. In particular, every prime-to- p isogeny is liftable.*

Proof. Suppose we are given a lifting

$$\tilde{f}[p^\infty]: \mathcal{G}_X \rightarrow \mathcal{G}_Y$$

of the isogeny of p -divisible groups $f[p^\infty]: X[p^\infty] \rightarrow Y[p^\infty]$. Then we can apply Lemma 6.2.1 to get a formal lifting of f to $\text{Spec}(V)$:

$$\tilde{f}: \mathcal{X} \rightarrow \mathcal{Y},$$

such that \tilde{f} is finite and \mathcal{Y} admits an algebraization. It suffices to show \tilde{f} also admits an algebraization, which can be done by [26, Proposition (5.4.4)].

The required lifting of p -divisible groups can be constructed as follows. Since $f[p^\infty]$ is separable, its kernel is a finite étale group scheme, which is freely liftable. Therefore, we may assume that $f[p^\infty]$ is an isomorphism. Let us fix a lifting of \mathcal{X} to V . The p -divisible group $\mathcal{G}_X := \mathcal{X}[p^\infty]$ over V forms a lifting of $X[p^\infty]$ to V , and such lifting gives a filtered Dieudonné module structure on $\mathbb{D}(Y[p^\infty])$ under the isomorphism $f[p^\infty]$. Then the statement follows from the Grothendieck–Messing theory (see the proof of [36, Proposition A.6] for example). \square

6.3. Specialization of derived isogenies. Next, we shall show that prime-to- p geometric derived isogenies are preserved under reduction. The basic idea is to show that two smooth projective family of abelian or K3 surfaces over a complete discrete valuation ring whose geometric generic fibers are Fourier–Mukai partners will have special fibers which are moduli space of stable twisted sheaves on each other. For this, we only need to specialize the datum that form a moduli space.

Theorem 6.3.1. *Let V be a discrete valuation ring with residue field k and let η be its generic point. Assume that $\text{char}(k) = p > 2$. Let $\mathcal{X} \rightarrow \text{Spec}(V)$ and $\mathcal{Y} \rightarrow \text{Spec}(V)$ be two projective families of abelian surfaces or K3 surfaces over $\text{Spec}(V)$. If there is a derived equivalence*

$$\Psi^P: D^b(\mathcal{X}_{\bar{\eta}}, \alpha) \xrightarrow{\sim} D^b(\mathcal{Y}_{\bar{\eta}}, \beta)$$

between geometric generic fibers such that $\text{ord}(\alpha)$ and $\text{ord}(\beta)$ are prime-to- p , then their special fibers are twisted derived equivalent.

Proof. We denote by X_0 and Y_0 the geometric special fibers of \mathcal{X}/V and \mathcal{Y}/V respectively. From Theorem 3.4.5, it is known that there is an isomorphism

$$\mathcal{Y}_{\bar{\eta}} \cong M_{\mathcal{H}_{\bar{\eta}}}^\alpha(\mathcal{X}_{\bar{\eta}}, v_{\bar{\eta}}),$$

for some twisted Mukai vector $v_{\bar{\eta}} \in \tilde{N}(\mathcal{X}_{\bar{\eta}}, \alpha)$, after replacing (\mathcal{X}, α) by $(\hat{\mathcal{X}}, \hat{\alpha})$ if it is necessary. Up to taking a finite extension of V , we may assume that the Brauer class α can be defined over η .

We claim that one can extend α to a class in the Brauer group of the total space $\text{Br}(\mathcal{X})$ if $p \nmid \text{ord}(\alpha)$. For simplicity, we assume $\text{ord}(\alpha) = \ell^n$ for some prime ℓ . In this case, the Gysin sequence and Gabber’s absolute purity gives an exact sequence

$$0 \rightarrow \text{Br}(\mathcal{X})\{\ell\} \rightarrow \text{Br}(\mathcal{X}_\eta)\{\ell\} \rightarrow \varinjlim_n H_{\text{ét}}^1(X_0, \mathbb{Z}/\ell^n). \quad (6.3.1)$$

(cf. [15, Theorem 3.7.1 (ii)]). If \mathcal{X} is a K3 surface, then we have $H_{\text{ét}}^1(X_0, \mathbb{Z}/\ell^n) = 0$ for all n as it is simply-connected, and thus one can find a lift $\tilde{\alpha} \in \text{Br}(\mathcal{X})$ of α by (6.3.1). When \mathcal{X} is an abelian surface over $\text{Spec}(V)$, the Gysin sequence can not directly give the existence of an extension of α . Again, one can use the trick of Kummer surfaces. Consider the relative Kummer surface $\text{Km}(\mathcal{X}) \rightarrow \text{Spec}(V)$, we have a commutative diagram

$$\begin{array}{ccc} \text{Br}(\text{Km}(\mathcal{X}))\{\ell\} & \xrightarrow{\sim} & \text{Br}(\text{Km}(\mathcal{X}_\eta))\{\ell\} \\ \downarrow \Theta & & \downarrow \Theta_\eta \\ \text{Br}(\mathcal{X})\{\ell\} & \xrightarrow{\quad} & \text{Br}(\mathcal{X}_\eta)\{\ell\} \end{array}$$

from Proposition 2.1.1. After passing to a finite extension, we can assume α lies in the image of Θ_η . As the top arrow is surjective and Θ_η is an isomorphism, we may find an extension $\tilde{\alpha} \in \text{Br}(\mathcal{X})\{\ell\}$ whose restriction on \mathcal{X}_η is α .

As the family $\mathcal{X} \rightarrow \text{Spec}(V)$ is smooth and proper, the relative Picard scheme $\text{Pic}_{\mathcal{X}/V}$ is proper. Now, under the specialization of the Picard group

$$\text{Pic}(\mathcal{X}_\eta) \xleftarrow{\sim} \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{X}_0),$$

we can pick extensions $v \in \tilde{N}(\mathcal{X})$ and $\mathcal{H} \in \text{Pic}(\mathcal{X})$ of v_η and \mathcal{H}_η , so that $v|_{\mathcal{X}_\eta} = v_\eta$ and $\mathcal{H}|_{\mathcal{X}_\eta} = \mathcal{H}_\eta$. In general, the line bundle $H_0 = \mathcal{H}_k$ on the special fiber is not ample. However, we may replace \mathcal{H} by $\mathcal{H} \otimes \mathcal{M}^{\otimes n}$ for a relative ample line bundle on \mathcal{X}/V and $n \gg 0$, such that \mathcal{H}_0 and \mathcal{H}_η are both ample and v -generic (i.e., not lie in a wall of the Mukai vector v), since the

v -walls in the ample cones of $\mathcal{X}_{\bar{\eta}}$ and \mathcal{X}_0 are known to be (locally) finitely many hyperplanes (cf. [66, Proposition 3.10] for $\text{char}(k) = 0$ or [7, Proposition 4.1.14] for $\text{char}(k) = p$). Then we let $M_{\mathcal{H}}^{\tilde{\alpha}}(\mathcal{X}, v)$ be the corresponding relative moduli space of \mathcal{H} -stable twisted sheaves, which is smooth over V . The generic fiber of $M_{\mathcal{H}}^{\tilde{\alpha}}(\mathcal{X}, v) \rightarrow \text{Spec}(V)$ is isomorphic to $\mathcal{Y}_{\bar{\eta}}$ after a finite base extension since it is geometrically birational to $\mathcal{Y}_{\bar{\eta}}$ by the wall-crossing (see [46] for example).

Set $\alpha_0 = \tilde{\alpha}|_{X_0} \in \text{Br}(X_0)$. Note that its special fiber is also isomorphic to $M_{H_0}^{\alpha_0}(X_0, v_0)$ after some finite field extension, we have the following commutative diagram after taking a finite ring extension of V :

$$\begin{array}{ccccccc} M_{\mathcal{H}}^{\tilde{\alpha}}(\mathcal{X}, v) & \leftarrow & M_{\mathcal{H}_{\eta}}^{\alpha}(\mathcal{X}_{\eta}, v_{\eta}) & \xrightarrow{\cong} & \mathcal{Y}_{\eta} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(V) & \longleftarrow & \text{Spec}(k(\eta)) & \longrightarrow & \text{Spec}(k(\eta)) & \longrightarrow & \text{Spec}(V) \end{array}$$

According to Matsusaka–Mumford [43, Theorem 1], the isomorphism can be extended to the special fiber. Thus Y_0 is isomorphic to $M_{H_0}^{\alpha_0}(X_0, v_0)$. It follows that $D^b(X_0, \alpha_0) \simeq D^b(Y_0, \beta_0)$. \square

Using Remark 5.1.8, one can easily deduce that every prime-to- p derived isogeny can be specialized.

Remark 6.3.2. In Theorem 6.3.1, the original derived equivalence $\Psi^{\mathcal{P}}$ is shown to be extended to the whole family. We only replaced it by a Fourier–Mukai transform given by the universal family of twisted sheaves which is naturally be specialized.

Remark 6.3.3. Our proof fails when k is imperfect and the twisted derived equivalence is not prime-to- p . This is because if the associated Brauer class α has order p^n , the map

$$\text{Br}(\mathcal{X})[p^n] \rightarrow \text{Br}(\mathcal{X}_{\eta})[p^n]$$

may not be surjective (cf. [55, 6.8.2]).

6.4. Proof of Theorem 1.4.1. When X or Y is supersingular, the assertion will be proved in Proposition 6.6.8. So we can assume that X and Y both have finite height.

Proof of (i') \Rightarrow (ii'). We can assume the prime-to- p derived isogeny is defined over a finitely generated subfield of \bar{k} . By the definition of prime-to- p derived isogeny, we have an isomorphism of F -crystals

$$H_{\text{crys}}^2(X/W) \xrightarrow{\sim} H_{\text{crys}}^2(Y/W).$$

With p -adic Shioda's trick in Corollary 4.6.4, we can conclude that X and Y are prime-to- p isogenous. It remains to see they are principally isogenous. The easiest way for proving this is by lifting to characteristic 0.

As the composition of prime-to- p isogenies remains prime-to- p , it suffices to consider a single derived equivalence

$$D^b(\mathcal{X}) = D^b(X, \alpha) \simeq D^b(Y, \beta) = D^b(\mathcal{Y})$$

for some gerbes $\mathcal{X} \rightarrow X$ and $\mathcal{Y} \rightarrow Y$ satisfying that the orders of $\alpha = [\mathcal{X}]$ and $\beta = [\mathcal{Y}]$ are both prime-to- p . As shown in Theorem 3.4.5, we may assume that \mathcal{Y} is the moduli stack of \mathcal{X} -twisted coherent sheaves on X . Therefore, we can liftings \mathcal{X} and \mathcal{Y} respectively, to some finite extension W' of W whose generic fibers are derived equivalent: There exists \mathbb{G}_m -gerbes \mathfrak{X}/W' and \mathfrak{Y}/W' as liftings of \mathcal{X} and \mathcal{Y} such that \mathfrak{Y} is the relative moduli stack of \mathfrak{X} -twisted coherent sheaves over W' , and thus the universal sheaf induces a twisted derived equivalence between their generic fibers, which is prime-to- p by Lemma 6.1.1.

By Theorem 1.2.1, the generic fibers of the \mathcal{X} and \mathcal{Y} are prime-to- p principally isogenous over a finite extension K of $k(\eta)$. Suppose $f_K: \mathcal{X}_K \rightarrow \mathcal{Y}_K$ is a principal isogeny. The Néron extension property of smooth models \mathcal{X}, \mathcal{Y} ensures that the f_K extends to an isogeny $f: \mathcal{X} \rightarrow \mathcal{Y}$, and there is an isogeny $g: \mathcal{Y} \rightarrow \mathcal{X}$ such that $g \circ f = \text{deg}(f_K)$. Thus the restriction f_k over the special fibers is still a principal isogeny. Then we can conclude the special fibers are prime-to- p

principally isogenous by using Tate's spreading theorem for p -divisible groups (cf. [61, Theorem 4]).

Proof of (ii') \Rightarrow (i'). Suppose that we are given an isogeny $\phi: Y \rightarrow X$, which is prime-to- p of degree d^2 . By Proposition 6.2.2, we can lift it to a prime-to- p isogeny of degree d^2 over some finite flat extension V of W :

$$\Phi: \mathcal{Y} \rightarrow \mathcal{X}.$$

The isogeny Φ_K between the generic fibers induces a G_K -equivariant isometry

$$\frac{\Phi_K^*}{d}: H_{\text{ét}}^2(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p) \xrightarrow{\sim} H_{\text{ét}}^2(\mathcal{Y}_{\bar{K}}, \mathbb{Z}_p).$$

By Theorem 5.2.5, there exists a geometric prime-to- p derived isogeny $D^b(\mathcal{X}_K) \sim D^b(\mathcal{Y}_K)$ whose p -adic cohomological realization is $\frac{\Phi_K^*}{d}$. The assertion then follows from Theorem 6.3.1.

6.5. Further remarks. From the proof of Theorem 1.4.1 (i') \Rightarrow (ii'), we can see that the lifting-specialization argument also works for non prime-to- p derived isogenies. Thus we have

Theorem 6.5.1. *Suppose X and Y are abelian surfaces over \bar{k} with finite height and $\text{char}(k) \neq 2$. If X and Y are derived isogenous, then they are quasi-liftable principal isogenous.*

Moreover, we believe that the converse of Theorem 6.5.1 also holds.

Conjecture 6.5.2. *With the same assumptions in Proposition 6.5.1. Then X and Y are derived isogenous if and only if they are quasi-liftable isogenous.*

Our approach remains valid once there is a specialization theorem for non prime-to- p derived isogenies. According to the proof of Theorem 6.3.1, it suffices to know the existence of specialization of Brauer classes of order p . Following the notations in Theorem 6.3.1, this means that the restriction map $\text{Br}(\mathcal{X}) \rightarrow \text{Br}(\mathcal{X}_\eta)$ is surjective. See Remark 6.3.3.

Now we discuss the connections between the derived isogenies of abelian surfaces and their associated Kummer surfaces. Using the lifting argument, the following theorem is an immediate consequence of the known result in characteristic 0.

Theorem 6.5.3. *With the assumption as in Theorem 6.5.1. If X and Y are prime-to- p derived isogenous, then the associated Kummer surfaces $\text{Km}(X)$ and $\text{Km}(Y)$ are prime-to- p derived isogenous. Moreover, if two twisted surfaces $(\text{Km}(X), \alpha)$ and $(\text{Km}(Y), \beta)$ are derived equivalent with $p \nmid \text{ord}(\alpha)$ and $p \nmid \text{ord}(\beta)$, then X and Y are prime-to- p derived isogenous.*

Proof. For the first assertion, as before, we can quasi-lift the prime-to- p derived isogeny between X and Y to characteristic 0. By Theorem 1.4.1 and Lemma 6.1.1, their liftings are geometrically prime-to- p derived isogenous. According to [60, Corollary 4.3], we get that the associated Kummer surfaces are prime-to- p derived isogenous. It follows from Theorem 6.3.1 that $\text{Km}(X)$ and $\text{Km}(Y)$ are prime-to- p derived isogenous.

For the last assertion, according to [9, Theorem 5.8], we can find liftings

$$(\mathcal{S}_1, \tilde{\alpha}) \rightarrow \text{Spec } W', (\mathcal{S}_2, \tilde{\beta}) \rightarrow \text{Spec } W'$$

of $(\text{Km}(X), \alpha)$ and $(\text{Km}(Y), \beta)$ over discrete valuation ring W' with residue field k and fraction field K' such that

- (1) the generic fibers $(\mathcal{S}_{1,K'}, \tilde{\alpha}|_{K'})$ and $(\mathcal{S}_{2,K'}, \tilde{\beta}|_{K'})$ are geometrically derived equivalent.
- (2) $\text{NS}(\mathcal{S}_{1,K'}) \cong \text{NS}(\text{Km}(X))$ and $\text{NS}(\mathcal{S}_{2,K'}) \cong \text{NS}(\text{Km}(Y))$ via the specialization map,

As seen in the proof of Lemma 2.2.4, with condition (2), we know that \mathcal{S}_1 is isomorphic to $\text{Km}(\mathcal{X})$ and \mathcal{S}_2 is isomorphic to $\text{Km}(\mathcal{Y})$ for some liftings of X and Y respectively. By Theorem 1.2.1, the generic fibers of \mathcal{X} and \mathcal{Y} are geometrically prime-to- p derived isogeny. Again, the assertion follows from Theorem 6.3.1. \square

Remark 6.5.4. It is natural to ask if one can apply the lifting method to prove the converse of Theorem 6.5.3, i.e. if $\mathrm{Km}(X)$ and $\mathrm{Km}(Y)$ are prime-to- p derived isogenous, so is X and Y . The issue is that the derived isogeny between $\mathrm{Km}(X)$ and $\mathrm{Km}(Y)$ is only quasi-liftable, not liftable. In other words, although we can lift every derived equivalence between twisted abelian surfaces or K3 surface to characteristic 0, we can not necessarily find some liftings of X and Y respectively such that the generic fibers of their associated Kummer surfaces are prime-to- p geometrically derived isogenous.

6.6. Supersingular twisted abelian surfaces. At last, we come to the case X is supersingular over an algebraically closed field k such that $\mathrm{char}(k) = p > 2$, i.e., X is isogenous to $E \times E$, where E is a supersingular elliptic curve over k .

6.6.1. supersingular twisted derived Torelli theorem. We have the following observation via Ogus's supersingular Torelli theorem [50, Theorem 6.2].

Theorem 6.6.1. *Let X and Y be two supersingular abelian surfaces over k . For \mathbb{G}_m -gerbes $\mathcal{X} \rightarrow X$ and $\mathcal{Y} \rightarrow Y$, the following statements are equivalent:*

- (1) *There is a Fourier-Mukai transform $D^b(\mathcal{X}) \simeq D^b(\mathcal{Y})$.*
- (2) *There is an isomorphism between K3 crystals $\tilde{H}(\mathcal{X}, W) \cong \tilde{H}(\mathcal{Y}, W)$.*

Proof. The proof is similar to the case of K3 surfaces which is given in [6, Theorem 3.5.5]. We sketch it here.

For (1) \Rightarrow (2), we only need to show the twisted crystalline Chern character of a Fourier-Mukai kernel \mathcal{P} lies $\tilde{H}(\mathcal{X}^{-1} \times \mathcal{Y}, W)$. When $p > 3$ as 2 and 3 are the only primes dividing the denominators in the formula of twisted Chern character of \mathcal{P} , this is due to a direct Chern character computation (cf. [6, Proposition A.3.3]). When $p = 3$, one can follow [7, Proposition 4.2.4] using the twistor lines and lifting argument. As the proof is similar, we omit the details here.

To prove that (2) implies (1), let us take $v = \rho(0, 0, 1)$, there is a filtered isomorphism

$$\gamma: \tilde{H}(\mathcal{X}, W) \xrightarrow{\rho} \tilde{H}(\mathcal{Y}, W) \xrightarrow{\phi^\mathcal{E}} \tilde{H}(\mathcal{M}_H(\mathcal{Y}, v), W) \quad (6.6.1)$$

where $\phi^\mathcal{E}$ is the cohomological Fourier-Mukai transform induced by the universal twisted sheaf \mathcal{E} on $Y \times \mathcal{M}_H(\mathcal{Y}, v)$. Then there is an isomorphism

$$f: X \xrightarrow{\sim} M_H(\mathcal{Y}, v)$$

since γ induces an isomorphism between supersingular K3 crystals (cf. (3.3.1))

$$H_{\mathrm{crys}}^2(X/W) \xrightarrow{\sim} H_{\mathrm{crys}}^2(Y/W).$$

The equality $[\mathcal{X}] = f^*[\mathcal{M}_H(\mathcal{Y}, v)]$ is from the construction. \square

6.6.2. twistor space of supersingular abelian surfaces. In [7], Bragg and Lieblich have developed the theory of twistor space for supersingular K3 surfaces. In terms of it, they are able to constructed the twisted period space of supersingular K3 surfaces. One can recap their construction and extend it to abelian surfaces as below. Firstly, we need the representability of flat cohomology of abelian surfaces, which plays an important role in the construction of supersingular twistor space.

Let $f: X \rightarrow S$ be a flat and proper morphism of schemes of finite type over k . Consider the sheaf of abelian groups $R^i f_* \mu_p$ on the big fppf site $(\mathrm{Sch}/S)_{\mathrm{fl}}$, which can be expressed as the fppf-sheafification of

$$S' \mapsto H_{\mathrm{fl}}^i(X_{S'}, \mu_p)$$

for any S -scheme S' . In general, the representability of $R^i f_* \mu_p$ is not easy to see by the ‘‘wildness’’ of flat cohomology with p -torsion coefficients. In this part, we will prove the representability for supersingular abelian surfaces.

Suppose S is perfect. Consider the auxiliary big fppf site $(\mathbf{Perf}/S)_{\text{fl}}$ for the full subcategory $\mathbf{Perf}/S \subset \mathbf{Sch}/S$ whose objects are perfect schemes over S . There is a functor between category of flat sheaves

$$(-)^{\text{perf}}: \text{Sh}((\mathbf{Sch}/S)_{\text{fl}}) \rightarrow \text{Sh}((\mathbf{Perf}/S)_{\text{fl}}). \quad (6.6.2)$$

induced by the natural inclusion $(\mathbf{Perf}/S)_{\text{fl}} \hookrightarrow (\mathbf{Sch}/S)_{\text{fl}}$, called *perfection*.

Proposition 6.6.2. *Let $f: X \rightarrow S$ be an abelian S -scheme of relative dimension 2, whose geometric fibers are all supersingular. Then*

- (1) $R^1 f_* \mu_p \cong \widehat{X}[p]$ is a finite flat S -group scheme whose geometric fibers are of local-local type (i.e. being self-dual under Cartier duality).
- (2) For any $\pi: \text{Spec}(A) \rightarrow S$ with A being perfect, we have $H_{\text{fl}}^i(A, \pi^* R^1 f_* \mu_p) = 0$ for $i \geq 1$. In particular, if S is perfect, then $(R^1 f_* \mu_p)^{\text{perf}} = 0$.

Proof. For (1), it suffices to check them affine locally on the base. Assume S is an affine scheme of finite type over k . By taking the Stein factorization, we can further assume $f_* \mathcal{O}_X \cong \mathcal{O}_S$. Then $f_* \mu_p \cong \mu_p$ also holds universally. Under this assumption, we have an exact sequence of fppf-sheaves by Kummer theory:

$$0 \rightarrow R^1 f_* \mu_p \rightarrow R^1 f_* \mathbb{G}_m \rightarrow R^1 f_* \mathbb{G}_m. \quad (6.6.3)$$

Since $R^1 f_* \mathbb{G}_m$ computes the relative Picard scheme $\text{Pic}_{X/S}$ and the Néron-Severi group of X is torsion-free, we can see

$$R^1 f_* \mu_p \cong \ker \left(\text{Pic}_{X/S} \xrightarrow{p} \text{Pic}_{X/S} \right) \cong \ker \left(\text{Pic}_{X/S}^0 \xrightarrow{p} \text{Pic}_{X/S}^0 \right).$$

On the other hand, it is well-known that $\text{Pic}_{X/S}^0$ is represented by the dual abelian S -scheme \widehat{X} (cf. [49, Corollary 6.8]). Thus $R^1 f_* \mu_p$ is representable by the commutative finite group S -scheme $\widehat{X}[p]$. Recall that Ogus's supersingular Torelli theorem implies that any supersingular abelian variety over an algebraically closed field is principally polarized, i.e., there is an isomorphism $X_t \cong \widehat{X}_t$ at all geometric fiber X_t (cf. [50, Corollary 6.15]). Thus the geometric fibers of $R^1 f_* \mu_p$ is of local-local type.

For (2), take the following smooth group resolution of α_p ,

$$0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \rightarrow 0,$$

we can see that $H_{\text{fl}}^i(A, \alpha_p) = 0$ for $i \geq 2$ for any ring A . For any finite flat group scheme G of local-local type, we can fill it in an exact sequence

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$$

such G' and G'' are of smaller p -ranks i.e., the rank of $\mathbb{Z}/p\mathbb{Z}$ -module $G'[p](\bar{k})$ and $G''[p](\bar{k})$ are less than $G[p](\bar{k})$. Thus by induction, we can prove that $H_{\text{fl}}^i(A, G) = 0$ for $i \geq 2$ and any finite flat group scheme G of local-local type.

For $i = 1$ and A being perfect, we can see

$$H_{\text{fl}}^1(A, \alpha_p) = A/A^p = 0.$$

Thus $H_{\text{fl}}^1(A, R^1 f_* \mu_p) = 0$ by the same induction as before. \square

Proposition 6.6.3. *Let $f: \mathcal{X} \rightarrow S$ be a smooth and proper family of supersingular abelian surfaces over an algebraic space S . Then $R^2 f_* \mu_p$ is representable by an algebraic space, which is separated and locally of finite presentation over S .*

Proof. This is a consequence of [8, Corollary 5.8, Example 5.9] as $R^1 f_* \mu_p$ is representable by Proposition 6.6.2. \square

Remark 6.6.4. The case that $X \rightarrow S = \text{Spec}(k)$ being a smooth surface for some field k is claimed by Artin in [2, Theorem 3.1] without proof. Bragg and Olsson also provide it a proof (Corollary 1.4 in loc. cit.)

For relative K3 surfaces, there is a moduli-theoretic proof given by Bragg and Lieblich using the stack of Azumaya algebras (*cf.* [7, Theorem 2.1.6]). Their proof can not be directly used for relative abelian surfaces as the essential assumption $R^1 f_* \mu_p = 0$ fails in fppf site $(\text{Sch}/S)_{\text{fl}}$.

Remark 6.6.5. An alternative proof for Proposition 6.6.3 is to apply Artin's representability criterion [1, Theorem 5.3]. The most hard part is to see the separatedness, which can be proved by (1) showing $(R^2 f_* \mu_p)^{\text{perf}}$ is representable by a separated algebraic space, and (2) running fpqc descent for the diagonal of $R^2 f_* \mu_p$.

The following observation is essential in the construction of twistor space of supersingular abelian or K3 surfaces.

Corollary 6.6.6 ([7, Proposition 2.2.4]). *Keep the assumptions same as in Proposition 6.6.3. The connected components of any geometric fiber of $R^2 f_* \mu_p \rightarrow S$ is isomorphic to the additive group scheme \mathbb{G}_a .*

Proof. Note that the completion of each geometric fiber of $R^2 f_* \mu_p$ at $s \in S$, along the identity section, is isomorphic to the formal Brauer group $\widehat{\text{Br}}_{X_s/k(s)}$, which is isomorphic to $\widehat{\mathbb{G}}_a$. The only smooth group scheme at $k(s)$ with this property is \mathbb{G}_a . \square

Now, let us recap the constructions of supersingular twisted period space and period morphism given in [7] and extend them to supersingular abelian surfaces. Fix a supersingular K3 lattice Λ , which is a free \mathbb{Z} -lattice whose discriminant $\text{disc}(\Lambda \otimes \mathbb{Q}) = -1$, signature $(1, n)$ ($n = 5$ or 21) and the Λ^\vee/Λ is p -torsion. Then $|\Lambda^\vee/\Lambda| = p^{2\sigma_0(\Lambda)}$ for $1 \leq \sigma_0(\Lambda) \leq \frac{(n-1)}{2}$. The lattice Λ is also determined by $\sigma_0(\Lambda)$, called the *Artin invariant* of Λ . Set

$$\tilde{\Lambda} = \Lambda \oplus U(p) \text{ and } \tilde{\Lambda}_0 = p\tilde{\Lambda}^\vee/p\tilde{\Lambda}.$$

where $U(p)$ is the twisted hyperbolic plane generated by vectors e and f such that $e^2 = f^2 = 0$ and $e \cdot f = -p$.

Let M_{Λ_0} be the moduli space of characteristic subspaces of $\Lambda_0 := p\Lambda^\vee/p\Lambda \cong e^\perp/e$ as in [51, §4] and let $M_{\Lambda_0}^{(e)}$ be the moduli space of characteristic subspaces of $\tilde{\Lambda}_{K3,0}$ which don't contain e (*cf.* [7, Definition 3.1.7]). They are both defined over \mathbb{F}_p .

For any $\tilde{K} \in M_{\Lambda_0}^{(e)}(S)$ over a \mathbb{F}_p -scheme S , one can produce a characteristic subspace K of $\Lambda_0 \otimes \mathcal{O}_S$ as the image of $\tilde{K} \cap (e^\perp \otimes \mathcal{O}_S)$ in $\Lambda_0 \otimes \mathcal{O}_S \cong e^\perp/e \otimes \mathcal{O}_S$ (*cf.* [7, Lemma 3.19]). Then the assignment $\tilde{K} \mapsto K$ gives a morphism

$$\pi_e: M_{\Lambda_0}^{(e)} \rightarrow M_{\Lambda_0}.$$

The Lemma 3.1.15 in [7] shows that the fiber of π_e at a k -point $[K] \in M_{\Lambda_0}(k)$ is isomorphic to a group scheme with connected components \mathbb{A}^1 .

Definition 6.6.7. The twistor line in $\overline{M}_{\tilde{\Lambda}_0} := M_{\tilde{\Lambda}_0} \times_{\mathbb{F}_p} k$ is an affine line $\mathbb{A}^1 \subset \overline{M}_{\tilde{\Lambda}_0}$ which is a connected component of a fiber of π_e over a k -point $[K] \in M_{\Lambda_0}(k)$ for some isotropic $e \in \tilde{\Lambda}_0$.

To emphasis that we are at either the case $n = 21$ or $n = 5$, we may write $\Lambda = \Lambda_{K3}$ and $\Lambda = \Lambda_{Ab}$ respectively. For K3 surfaces, it has been shown that the moduli functor $S_{\Lambda_{K3}}$ of Λ_{K3} -marked supersingular K3 is representable by a locally of finite presentation, locally separated and smooth algebraic space of dimension $\sigma_0(\Lambda_{K3}) - 1$. There is a universal family

$$u: \mathcal{X} \rightarrow S_{\Lambda_{K3}}$$

(as algebraic spaces), which is smooth with relative dimension 2. The higher direct image $R^2 u_*^{\text{fl}} \mu_p$ is representable by an algebraic group space over $S_{\Lambda_{K3}}$ after perfection, denoted by

$$\pi: \mathcal{S}_{\Lambda_{K3}} \rightarrow S_{\Lambda_{K3}}$$

(see loc.cit. Theorem 2.1.6). The connected component of the identity $\mathcal{S}_{\Lambda_{K3}}^o \subset \mathcal{S}_{\Lambda_{K3}}$ parameterizes the μ_p -gerbes which are not essentially-trivial except the identity, at each Λ -marked K3 surface in $S_{\Lambda_{K3}}(k)$. Then there are (*twisted*) *period morphisms*

$$\rho : S_{\Lambda_{K3}} \rightarrow M_{\Lambda_{K3,0}} \text{ and } \tilde{\rho} : \mathcal{S}_{\Lambda_{K3}}^o \rightarrow M_{\Lambda_0}^{(e)},$$

(cf. [51, §3] and [7, Definition 3.5.7]). Then the method in *loc.cit.* shows that there is a Cartesian diagram

$$\begin{array}{ccc} \mathcal{S}_{\Lambda_{K3}}^o & \xrightarrow{\pi} & S_{\Lambda_{K3}} \\ \downarrow \tilde{\rho} & & \downarrow \rho \\ M_{\Lambda_{K3,0}}^{(e)} & \longrightarrow & M_{\Lambda_{K3,0}} \end{array} \quad (6.6.4)$$

and ρ is étale surjective. The twisted period map $\tilde{\rho}$ factors as

$$\begin{array}{ccccc} & & \tilde{\rho} & & \\ & \searrow & \curvearrowright & \searrow & \\ \mathcal{S}_{\Lambda_{K3}}^o & \xrightarrow{\tilde{\rho}'} & \mathcal{P}_{\Lambda_{K3}} & \longrightarrow & M_{\Lambda_{K3,0}}^{(e)}, \end{array}$$

where $\mathcal{P}_{\Lambda_{K3}}$ is the moduli space of ample cones of characteristic subspaces defined by Ogus ([51]). It has been shown that $\tilde{\rho}'$ is an isomorphism (cf. [7, Theorem 5.1.7]). In particular, this implies that the moduli space of supersingular K3 surfaces of Artin invariant ≤ 2 is rationally fibered over the moduli space of supersingular K3 surfaces of Artin invariant 1, whose fiber is a twistor line, corresponding to the relative moduli spaces of twisted sheaves on universal gerbes associated to the Brauer groups of the superspecial K3 surface.

For supersingular abelian surfaces, everything works by replacing Λ_{K3} with Λ_{Ab} . Indeed, the proof in [7, Proposition 5.1.5] already shows that the twisted period map $\tilde{\rho}'$ for abelian surfaces will be an isomorphism. Another simple way to see this is via the Kummer construction. One just notice that the moduli space of supersingular abelian surfaces is isomorphic to the moduli space of supersingular Kummer surfaces and they have isomorphic period spaces, i.e.,

$$M_{\Lambda_{K3,0}}^{(e)} \cong M_{\Lambda_{Ab,0}}^{(e)}$$

when $\sigma_0(\Lambda_{K3}) = \sigma_0(\Lambda_{Ab}) \leq 2$. This gives

Proposition 6.6.8. *For non-superspecial supersingular abelian surface X' , there exists a Brauer class $[\mathcal{X}] \in \text{Br}(X)$ such that $D^b(\mathcal{X}) \simeq D^b(X')$. In particular, X' is a moduli space of twisted sheaves on X .*

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