Introduction to Hodge Conjecture

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Introduction

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  ⋆ Hodge Theory
  ⋆ Hodge conjecture and its original conjecture
  ⋆ Examples and Counter examples
  ⋆ Hodge conjecture for abelian varieties

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  - ⋆ Hodge Theory
  - ⋆ Hodge conjecture and its original conjecture
  - ⋆ Examples and Counter examples
  - ⋆ Hodge conjecture for abelian varieties

- Goal of this seminar:
  - ⋆ Infinitesimal and generalized Hodge Conjecture
  - ⋆ Hodge conjecture for CM abelian varieties
  - ⋆ Markman's paper
  - ⋆ Tate conjecture

Cohomology Theories

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  has another structure depending on the field of definition.
  de-Rham Theorem: $\mathrm{H}^i(X,\mathbb{Z})_B \otimes \mathbb{R} \cong \mathrm{H}^i(X_{cl},\mathbb{R})$.

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- **Dolbeault cohomology**: $\mathrm{H}^{p,q}(X^{an}) = \mathrm{H}^q(X^{an},\Omega^p_X)$ as the sheaf cohomology.
  The Dolbeault resolution gives

$$\mathrm{H}^{p,q}(X^{an}) = \frac{\bar{\partial}\text{-closed } (p,q) \text{ differential forms}}{\bar{\partial}\text{-exact } (p,q) \text{ differential forms}}$$

Hodge decomposition

Hodge decomposition for a compact Kähler manifold (X, ω)

- $\Delta_d = dd^* + d^*d$ the Laplace operator
  $\mathcal{H}^{p,q}(X) =$ the set of classes of Δ_d-harmonic form of type (p, q)
  $\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}}^{q,p}(X)$.
- $\mathrm{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$, and the decomposition does not depend on the choice of the Kähler form ω.
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Hodge structure

- The decomposition of $\mathrm{H}^i(X, \mathbb{Z}) \otimes \mathbb{C}$ together with the Hodge symmetry is called the Hodge structure on $\mathrm{H}^i(X, \mathbb{Z})$.
- The Hodge structure above together with the bilinear form

$$(*, *) := * \cup * \cup \omega^{n-i}$$

  is called a polarized Hodge structure.

Hodge classes

Definition

- An integral Hodge class of degree $2k$ on X is an element α in the space

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- **Hodge Filtration**: $\mathrm{F}^i\mathrm{H}^m(X,\mathbb{C}) = \bigoplus_{p\geq i} \mathrm{H}^{p,q}(X)$ a decreasing filtration of $\mathrm{H}^m(X,\mathbb{C})$.
  Then $\alpha \in \mathrm{Hdg}^{2k}(X) \Leftrightarrow \alpha \in \mathrm{H}^{2k}(X,\mathbb{Q}) \cap \mathrm{F}^k\mathrm{H}^{2k}(X,\mathbb{C})$.

Example: fundamental class of subvarieties

- Let $i : Z \subseteq X$ be a smooth closed subvariety of codimension k, hence a real oriented manifold of dimension $2n - 2k$.

  There is a fundamental class

$$[Z] \in H_{2n-2k}(Z, \mathbb{Z})$$

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- More generally, the Chern classes of coherent sheaves are integral Hodge classes, i.e. $c_i(\mathcal{F}) \in \mathrm{Hdg}^{2i}(X, \mathbb{Z})$.

Hodge conjecture

- X: smooth projective variety over $\mathbb{C}$.
- We define

$$\mathcal{Z}^{2k}(X) = \frac{\mathrm{Hdg}^{2k}(X)}{< \sum_i [Z_i],\ Z_i \subseteq X >}$$

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- Hodge's original Conjecture, 1950'

  The group $\mathcal{Z}^{2k}(X)$ *is zero, i.e. every class in* $\mathrm{Hdg}^{2k}(X, \mathbb{Z})$ *is algebraic.*

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- Hodge Conjecture of degree 2k

  The group $\mathcal{Z}^{2k}(X)$ is torsion.

Degree 2: Lefschetz (1,1)-theorem

- On X^{an}, the exponential exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\exp} \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

induces a long exact sequence

$$\mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X) \to$$

- $\mathrm{H}^1(X, \mathcal{O}_X^\times) \cong$ the group of isomorphism classes of holomorphic line bundles.
- The kernel $\ker(\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X))$ is exactly $\mathrm{Hdg}^2(X, \mathbb{Z})$.

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Lefschetz-(1,1) Theorem

Every class in $\mathrm{Hdg}^2(X, \mathbb{Z})$ can be represented by a linear combination of divisor classes.

Failure of Hodge's original conjecture

- The integral Hodge conjecture fails for many reasons.
- **Atiyah-Hirzebruch-Totaro**: topological obstruction via cobordism. There exist torsion classes in $\mathrm{Hdg}^{2k}(X, \mathbb{Z})$, which can be not represented by algebraic classes.

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Theorem

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Theorem

Suppose that $p \neq 2, 3$ and $p^3 | d$. A generic hypersurface X of degree d in $\mathbb{P}^4$ does not satisfy the integral Hodge conjecture.

- $\exists X \to B$ a family of smooth projective varieties and a locally constant integral Hodge class
$$\alpha_t \in \mathrm{H}^4(X_t, \mathbb{Z})$$
with the property that on a dense subset $B_{alg} \subseteq B$, the class α_t is algebraic, but on its complementary set, the class α_t is not algebraic.

We say X admits a decomposition of diagonal if

$$N\Delta_X = Z_1 + Z_2 \in \mathrm{CH}_n(X \times X) \tag{1}$$

where $Z_1 \subseteq S \times X$ and $Z_2 \subseteq X \times W$ for some threefold $S \subseteq X$ and proper subvariety $W \subseteq X$.

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Theorem (Bloch & Srinivas)

If X admits a decomposition of diagonal as above, then the Hodge conjecture of degree 4 holds on X.

Examples for Hodge conjecture of degree 4

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An immediate corollary is

Corollary

Hodge conjecture of degree 4 holds on rationally connected varieties. In particular, it holds on Fano varieties.

Examples: algebraicity of Hodge isometry

- Let S be a K3 surface.
- A rational Hodge isometry

$$\varphi : \mathrm{H}^2(S_1, \mathbb{Q}) \cong \mathrm{H}^2(S_2, \mathbb{Q})$$

can be viewed as a Hodge class in $\mathrm{Hdg}^4(S_1 \times S_2)$ via Kunneth formula.

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Theorem (Buskin, Huybrechts)

Any Hodge isometry $\varphi : \mathrm{H}^2(S_1, \mathbb{Q}) \to \mathrm{H}^2(S_2, \mathbb{Q})$ between two Kähler K3 surfaces S_1 and S_2 is a polynomial in Chern classes of coherent analytic sheaves over $S_1 \times S_2$.

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- **Open problem**: show the Hodge isogenous

$$\varphi : \mathrm{H}^2(S, \mathbb{Q}) \to \mathrm{H}^2(S, \mathbb{Q})$$

is algebraic.

HC for abelian varieties

Let A be an abelian variety over $\mathbb{C}$.

Definition

- Abelian varieties A and A' are said to be isogeneous if there is a finite, surjective map : $A \to A'$. It preserves HC.
- A is simple if A is not isogenous to a product of abelian varieties.

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An overview of results

- (Mattuck) HC holds for general abelian varieties.
- (Tate) HC holds for a product of elliptic curves.
- (Tankeev) HC holds for abelian varieties whose dimension is a prime.

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- (Mumford) This fails for some simple abelian fourfold.

HC for abelian varieties: Weil type

- An abelian variety of Weil-type of dimension $2n$ is a pair (A, K) with A a $2n$ dimensional abelian variety and

$$K \to \mathrm{End}(A) \otimes \mathbb{Q}$$

is an imaginary quadratic field such that the action of K on the tangent space $T_0(A)$ can be diagonalized as

$$\mathrm{diag}(\sigma(k), \ldots, \sigma(k), \bar{\sigma}(k), \ldots, \bar{\sigma}(k))(k \in K).$$

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- A polarized abelian variety of Weil-type is a triple (A, K, L) with the polarization on $H^2(X, \mathbb{Z})$ as

$$L(kx, ky) = \sigma(k)\bar{\sigma}(k)L(x, y)$$

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- Set $K = \mathbb{Q}(\varphi)$, the Hermitian form $H(x, y) = L(\varphi x, y) + L(x, \varphi y)$ can be represented by a diagonal matrix $\mathrm{diag}(a, 1, \ldots, 1, -1, \ldots, -1)$. $a = |\det(H)| > 0$ is called the discriminant of (A, K, L).

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Theorem (Weil)

For a general $2n$-dimensional abelian variety X of Weil-type (with $n > 1$), one has $\mathrm{Hdg}^{2n}(X)$ is not generated by $\mathrm{NS}(X)$.

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Theorem (Moonen-Zarhin)

Let A be a simple abelian variety of dimension 4 with $\mathrm{Hdg}^4(X)$ not generated by $\mathrm{NS}(X)$. Then A is of Weil-type.

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Recent progress

Theorem (Markman)

Hodge conjecture holds for generic abelian fourfolds of Weil-type with trivial discriminant ($a = 1$).