Seminar2020-02: Examples of the integral Hodge conjecture

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1 Atiyah-Hirzebruch-Totaro topological obstruction

Let X be a smooth projective variety of real dimension $2n$ over $\mathbb{C}$. The set of its complex points $X(\mathbb{C})$ naturally forms a complex manifold. We will set

$$H^{2k}(X, \mathbb{Z}) \coloneqq H^{2k}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \tag{1}$$

in the following literature.

Exercise 1.1. Any torsion class $s \in H^{2k}(X, \mathbb{Z})$, i.e. there is some integer n such that $ns = 0 \in H^{2k}(X, \mathbb{Z})$, is a Hodge class.

As torsion classes in $H^{2k}(X, \mathbb{Z})$ are all Hodge classes, we may ask whether the torsion classes are all algebraic. Examples of non-algebraic torsion classes are firstly discovered by Atiyah and Hirzebruch, and revisited by Totaro. We will show in this section that the cycle map factors through a more refined group, the complex coboardism group.

Definition 1.1. A weakly complex manifold M is a real manifold with a complex structure on $TM \oplus \mathbb{R}^N$ for some integer N.

We identify two complex structure on $TM \oplus \mathbb{R}^N$ if they are equivalent in the complex K_0 group.

The complex boardism group MU_iX is defined to be the free abelian group generated by all continuous maps $M \to X$ from a closed weakly complex manifold M of dimension i to X, modulo the relations

$$[M_1 \amalg M_2 \to X] = [M_1 \to X] + [M_2 \to X]$$
$$[\partial M \to X] = 0$$

Definition 1.2. The complex coboardism is defined by $MU^iX = MU_{2n-i}X$ and $MU_* = MU_*(pt)$.

Note that MU_*X is a MU_* module via product and there is a map $\phi : MU_jX \to H_j(X)$ sending $[M \xrightarrow{f} X]$ to f_*M. By the module structure, $MU_{>0} \cdot MU_*X$ are kill by ϕ. So we get

$$\phi : MU_*X \otimes_{MU_*} \mathbb{Z} \to H_*(X, \mathbb{Z})$$

Theorem 1.3. *We can define a homomorphism from Z_iX to $MU_{2i}X \otimes_{MU_*} \mathbb{Z}$, such that the composition with ϕ is the usual cycle map.*

In fact, for a cycle $Z \subset X$, we can associate it to $[Z' \to Z \to X]$, where $Z' \to Z$ is a resolution of singularities. Then, the composition is the usual cycle map if $[Z' \to Z \to X]$ is well defined, i.e, independent of resolutions. It is guarenteed by the following theorem.

Theorem 1.4. *If X is a finite cell complex. Then $\phi : \phi : MU^*X \otimes_{MU^*} \mathbb{Z} \to H^*(X, \mathbb{Z})$ is an isomorphism in degree ≤ 2.*

Then, a torsion class not in the image of ϕ will be a counterexample of the integral Hodge conjecture. In fact, Atiyah and Hirzebruch constructed a projective variety X which is n-homotopic to product of some classfying space and satisfies the condition.

2 Kollár's example

In this section, We will introduce Kollár's counterexample, finding non-algebraic Hodge classes with some multiple algebraic.

Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Then, $H^4(X, \mathbb{Z}) = \mathbb{Z}\alpha$ with $\deg \alpha = 1$, by Lefschetz hyperplane theorem and Poincare duality. We will show that all curves on some X are of degree divided by a fixed integer $p > 1$. So α can't be algebraic.

Theorem 2.1. *Assume $(p, 6) = 1$, $d = p^3 s$. then for very general X and any curve $C \subset X$, we have $p|degC$*

Proof. We firstly find one singular X_0 satisfying the conclusion and then a deform very general hypersurface to X_0.

Let $Y \subset \mathbb{P}^4$ be a smooth hypersurface of degree s. We have the following

$$f : Y \subset \mathbb{P}^4 \overset{|\mathcal{O}_{\mathbb{P}^4}(p)|}{\longrightarrow} \mathbb{P}^N \overset{projection}{\longrightarrow} \mathbb{P}^4$$

and set $X_0 = f(Y)$, where the projection is general.

Estimating the dimensions, we find that f is generic finite, 2 to 1 over a surface, 3 to 1 over a curve and 4 to 1 over finite points.

Then, if C_0 is a curve on X_0, we can find a curve $C_0' \subset Y$, such that $f_* C_0' = 6C_0$. Thus we find $p|degC_0$ by projection formula and $(p, 6) = 1$.

Next, we deform a very general $C \subset X$ to $C_0 \subset X_0$. Let $\mathbb{P}^N$ be the projective space of all polynomials of degree d on $\mathbb{P}^4$, and let $\mathcal{X} \to \mathbb{P}^N$ be the universal hypersurface. Define the relative Hilbert schemes

$$\mathcal{H}_\nu \to \mathbb{P}^N,$$

parameterizing pairs $\{(C, X), Z \subset X\}$, where C is a curve with Hilbert polynomial ν. We know that there are only countably many Hilbert polynomials since they are determined by degree and genus.

we have the following :

- The morphism $g_\nu : \mathcal{H}_\nu \to \mathbb{P}^N$ is projective and $\amalg g_\nu$ dominates $\mathbb{P}^N$
- There exists a universal subscheme $\mathcal{Z}_\nu \subset \mathcal{H}_\nu \times_{\mathbb{P}^N} \mathcal{X}$ which is flat over $\mathcal{H}_\nu$.

Let $U = \mathbb{P}^N \setminus \bigcup_{\nu \in I} g_\nu(\mathcal{H}_\nu)$, where I is the set of ν such that g_ν is not surjective. U is nonempty since $\amalg g_\nu$ dominates $\mathbb{P}^N$. Therefore, for X in U, any curve $C \subset X$ can be deformed to a curve $C' \subset X_0$. Then the theorem follows by the flatness of the universal cycle.

□

3 Cubic fourfolds

In this section, we show that the integral Hodge conjecture is true for cubic fourfolds. Firstly, we recall some facts about intermediate Jacobian.

Let X be a smooth projective variety of dimension n. We have the Hodge decomposition:

$$H^{2k-1}(X,\mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}$$

So $H^{2k-1}(X,\mathbb{Z})$ can be seen as a lattice in $H^{2k-1}(X,\mathbb{C})/F^k H^{2k-1}(X)$ and we have the following definition:

Definition 3.1. The kth intermediate Jacobian of X ie the complex torus

$$J^{2k-1}(X) = H^{2k-1}(X,\mathbb{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z}))$$

If a cycle $Z \in Z^k(X)$ is homologous to 0, there is a differentiable chain $\Gamma \subset X$ of dimension $20 - 2k + 1$ such that $\partial\Gamma = Z$. Using the fact that Z is of deg (k,k) and the Stokes formula, we may get $\int_\Gamma \in H^{2k-1}(X,\mathbb{C})/F^k H^{2k-1}(X)$. Further, if $\partial\Gamma' = Z$ for another Γ', we can show that $\int_\Gamma - \int_{\Gamma'} \in H^{2k-1}(X,\mathbb{Z})$. Combining those, we get the so called the Abel-Jacobi map:

$$\begin{aligned} \Phi_X^k : Z^k(X)_{hom} &\to J^{2k-1}(X) \\ Z \quad &\mapsto \quad \int_\Gamma \end{aligned}$$

We introduce another way to define intermediate Jacobian, that is, Deligne cohomology. We define the Deligne complex $\mathbb{Z}_D(p)$ to be

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \Omega_X \to \cdots \to \Omega_X^{p-1} \to 0$$

and the Deligne cohomology $H_D^k(X,\mathbb{Z}(p))$ is just the hypercohomology of the Deligne complex.

We know the Deligne complex fits into a short exact sequence:

$$0 \to \Omega^{\leq p-1}[1] \to \mathbb{Z}_D(p) \to \mathbb{Z} \to 0$$

Run the corresponding long exact sequence and combine the fact that $H^k(X,\Omega^{\leq p-1}) = H^k(X,\mathbb{C})/F^p H^k(X)$, we get

$$0 \to J^{2p-1}(X) \to H_D^{2p}(X,\mathbb{Z}(p)) \to Hdg^{2p}(X,\mathbb{Z}) \to 0$$

The story is similar in relative version. Let $X \xrightarrow{\pi} B$ be a smooth projective morphism. Denote $\mathcal{H}_\mathbb{Z}^{2k-1} = R^{2k-1}\pi_*\mathbb{Z}$ and $F^p\mathcal{H}^{2k-1} = R^{2k-1}\pi_*\Omega_{X|B}^{\geq p}$.

Then, the intermediate Jacobian in family $\mathcal{J}^{2k-1} \to B$ is defined by:

$$0 \to \mathcal{H}_\mathbb{Z}^{2k-1} \to \mathcal{H}^{2k-1}/F^k\mathcal{H}^{2k-1} \to \mathcal{J}^{2k-1} \to 0$$

If $s \in H^0(B,\mathcal{J}^{2k-1})$ is a holomorphic section of $\mathcal{J}^{2k-1} \to B$, we get an element $\phi(s)$ vie the boundary map $\phi H^0(B,\mathcal{J}^{2k-1}) \to H^1(B,\mathcal{H}_\mathbb{Z}^{2k-1})$.

We need a theorem of Griffths:

Theorem 3.2. *if $Z \in Z^k(X)_{hom}$ is flat over B, wen can define a holomorphic morphism via Abel-Jacobi maps:*

$$\begin{aligned} \Phi_Z : B &\longrightarrow \mathcal{J}^{2k-1} \\ b &\longmapsto \Phi_{X_b}^k(Z_b) \end{aligned}$$

In particular, it shows that the Abel-Jacobi map factor through $CH^k(X)_{hom}$, since the image of the Abel-Jacobi map is contained in a abelian variety which do not contain any rational curve.

We now come back to a cubic fourfold Y. Since $H^6(Y, \mathbb{Z})$ is generated by a line and Y does contain a line, we only need to show the degree 4 case. The strategy is to associate a Hodge class $\alpha \in Hdg^4(Y, \mathbb{Z})$ a section of intermediate Jacobians in a certain family, and to show that the section comes from a algebraic cycle.

Let $\{X_t\}_{t\in\mathbb{P}^1}$ be a pencil of hyperplane sections with base locus a cubic surface S and X is the blow up of Y along S. We have:

$$\begin{array}{ccc} X & \xrightarrow{\tau} & Y \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^1 & & \end{array}$$

There exists a line $l \subset S$ and a integer d such that $\alpha|_{X_t} = dl$, since X_t and S are cubic threefold and surface. We can compute by blow up formula that $H^3(X, \mathbb{Z}) = 0$. Hence $J^3(X) = 0$

let $\beta = \tau^*\alpha - d[l \times \mathbb{P}^1]$. Then $\beta|_{X_t} = 0$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^3(X) & \longrightarrow & H^4_D(X, \mathbb{Z}(2)) & \longrightarrow & Hdg^4(X, \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle g_t} & & \downarrow{\scriptstyle h_t} & & \\ 0 & \longrightarrow & J^3(X_t) & \longrightarrow & H^4_D(X_t, \mathbb{Z}(2)) & \longrightarrow & Hdg^4(X_t, \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

The boundary map is $ker(h_t) \to coker(f_t)$. Since $\beta|_{X_t} = 0$ and $J^3(X) = 0$, we get a section $\varphi_\beta : B \to \mathcal{J}^3$. Zucker shows that this section is holomorphic and compatible with the Abel-Jacobi map. It is also showed that if another element in $ker(h_t)$ defines the same section with φ_β, they only differ by elements in fibres of $X \to \mathbb{P}^1$, which are algebraic.

Thus, it is enough to show that $\exists Z \in Z^2(X)_{hom}$, such that the section defined by Z as in theorem 5 is the same with φ_β.

We need the following theorem of Markushevitch and Tikhomirov:

Theorem 3.3. *The moduli space M_t of semi stable rank 2 torsion free sheaves with $c_1 = 0$ and $c_2 = 2l$ on X_t is birational to $J^3(X_t)$ via the Abel-Jacobi map $E \mapsto \Phi^2_X(c_2(E) - 2l)$.*

Then, we consider $M = \bigcup_{t\in\mathbb{P}^1} M_t$ and construct a object P over M parametrizing curves in X_t. We want to find some family of curves in P defining the same section with φ_β via the Abel-Jacobi map. Then, the surface swept out by the family of curves maps to class of $\alpha + kS$ for some k via τ and it implies that α is algebraic.

$P \to M$ is constructed as following. The fibre over $E_s \in M_t$ is $\mathbb{P}(H^0(X_t, E_s(k))$ for a sufficiently large k. For a general section in $H^0(X_t, E_s(k)$, its zero locus is a curve,. Hence P parametrizes curves in X_t and compatible with the Abel-Jacobi map.

Then, it is enough to show that the section φ_β can be lift to a section $B \to P$ via $P \to M \to \mathcal{J}^3$. It is due to M is birational to $\mathcal{J}^3$ and the Brauer group of a curve is trivial. That's complete the proof.